# BOSTON UNIVERSITY SOCIETY OF MATHEMATICS COLLOQUIUM PROCEEDINGS
# THE BRAID SYMMETRIES OF A DISK

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Abstract. A braid in math is not too far from how it sounds — a collection of strands interlacing and intertwining as they travel through paths in space. These braids are interesting creatures. Some braids look complex, but are just one twist from being untied; other braids can look simple, but are in truth much more obstinate. However, the study takes an unexpected turn upon discovering braids, like numbers, can be multiplied and divided to create new braids: they form an abstract structure called a group. By studying this group, we will realize that by studying braids, we are actually also studying the symmetries of a disk.

## 1. Preliminaries and Notation

In this paper, we will assume a working knowledge of basic set theory and the functions between them. The bare minimum assumptions on sets and functions are listed further below. Additionally, we will assume continuity of multivariable functions — an understanding on the level of a first multivariate calculus course should be enough to follow intuitively. Finally, we assume elementary knowledge of matrices and determinants, though this is purely contained in examples and informal discussions — it is not strictly necessary to understand the core content.

This is technically all the required prerequisites, but the reader without any previous exposure to group theory may find this paper unfairly demanding. Nonetheless, Section 2 gives a primer on all the group theory required from the beginning. We hope it is enough to fill in any gaps in knowledge.

Throughout this paper, many of the definitions and proofs will come with informal discussions alongside the full technical details. It is not necessary to fully understand every technical detail, only to glean some intuition. The formal proofs are generally not essential for the story, so feel free to skip them. The subject of this paper is very visual, and the core ideas can nearly always be grasped by intuition alone.

Readers are encouraged to skip any sections they are already familiar with.

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**Notation.**

| | |
|---|---|
| $\varnothing$ | The empty set |
| $s \in S$ | $s$ is an element of the set $S$ |
| $A \subset B$ | $A$ is a subset of $B$, or $a \in A$ implies $a \in B$ |
| $A \cup B$ | The union of $A$ and $B$ $\{s : s \in A \text{ or } s \in B\}$ |
| $A \cap B$ | The intersection of $A$ and $B$ $\{s : s \in A \text{ and } s \in B\}$ |
| $A \setminus B$ | The set difference of $A$ and $B$ $\{s : s \in A \text{ and } s \notin B\}$ |
| $A \times B$ | The cartesian product $\{(a, b) : a \in A \text{ and } b \in B\}$ |
| $S^n$ | The cartesian product $\{(s_1, s_2, \ldots, s_n) : s_j \in S\}$ |
| $f : A \to B$ | A function from the set $A$ to the set $B$ |
| $a \mapsto b$ | (of a function) maps the element $a$ to the element $b$ |
| $\mathrm{id} : A \to A$ | The identity function that takes each point of $A$ to itself. |

**Sets.**

| | |
|---|---|
| $\mathbb{Z}$ | The set of integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ |
| $\mathbb{R}$ | The set of real numbers |
| $D^n$ | The $n$-dimensional unit disk $\left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_j^2 \leq 1\right\}$ |
| $S^n$ | The $n$-dimensional unit sphere $\left\{(x_1, \ldots, x_{n+1}) \subset \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1)\right\}$. |

**Definition 1.1** (Properties of functions)**.** A function $f : A \to B$ between sets is **injective** if for all $a, a' \in A$ such that $a \neq a'$, $f(a) \neq f(a')$.

Likewise, $f$ is **surjective** if for all $b \in B$, there exists $a \in A$ such that $f(a) = b$. If $f$ is both injective and surjective, we say $f$ is **bijective**.

If there exists $g : B \to A$ such that for all $a \in A$ and $b \in B$,

$$(f \circ g)(b) = b \text{ and } (g \circ f)(a) = a,$$

we say $f$ is **invertible**.

**Theorem 1.2.** *Let $f : A \to B$ be a function between sets. Then $f$ is bijective if and only if $f$ is invertible.*

## 2. Introduction to Groups

**Definition 2.1** (Groups, [DF04, p. 16])**.**

**Informal.** We can think of groups as a way to generalize adding and multiplying to more abstract settings. For example, in the familiar situation of adding integers together, we can think of addition as a function whose input is an ordered pair $(a, b)$ and whose output is another integer we call $a + b$. We call this a *binary operation*, and the operation $+$ has the following nice properties in the integers:

(i) for all $a, b, c \in \mathbb{Z}$, $(a + b) + c = a + (b + c)$,
(ii) for all $a \in \mathbb{Z}$, $a + 0 = a = 0 + a$,
(iii) for all $a \in \mathbb{Z}$, $a + (-a) = 0 = (-a) + a$,
(iv) for all $a, b \in \mathbb{Z}$, $a + b = b + a$.

A group generalizes this idea to more abstract sets than the integers and more abstract operations than addition. For instance, we will see in Example 2.9 later that the symmetries of a triangle satisfy properties (i), (ii), and (iii). Further in the paper, we will see Mapping Class groups (Def. 5.6) and Braid groups (Def. 4.8). Any set with an operation that satisfies (i), (ii), and (iii), we call a *group* under that operation. If we further have property (iv), that group is called *abelian*.

**Formal.** A **group** is a pair $(G, *)$ where $G$ is a set and $*$ is a **binary operation**

$$* : G \times G \to G,$$

where we denote $*(g_1, g_2) = g_1 * g_2$, satisfying the **group axioms**:

(i) **Associativity**. For all $g_1, g_2, g_3 \in G$, $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$.
(ii) **Identity**. There exists $e \in G$ such for all $g \in G$, $e * g = g = g * e$. We call $e$ an **identity element** of $G$, or simply an identity.
(iii) **Inverses**. For all $g \in G$, there exists $h \in G$ such that $h * g = e = g * h$. We call $h$ an **inverse** of $g$.

A group that further satisfies

(iv) **Commutativity**. For all $g, h \in G$, $g * h = h * g$

is called **abelian**.

**Proposition 2.2** ([DF04, p. 18])**.** *If $(G, *)$ is a group, then*

*(i) The identity of $G$ is unique,*
*(ii) For each $g \in G$, the inverse of $g$ is uniquely determined.*

By Proposition 2.2, we can unambiguously denote the identity of a group $(G, *)$ by 1 and the inverse of an element $g \in G$ by $g^{-1}$. Then for any $n \in \mathbb{Z}$ and $g \in G$, we define

$$g^n = \begin{cases} g^n = 1 & \text{if } n = 0, \\ \underbrace{g * g * \cdots * g}_{n \text{ times}} & \text{if } n > 0, \\ (g^{-1})^{-n} & \text{if } n < 0. \end{cases}$$

If $(G, *)$ is abelian, we often choose the symbols 0, $-g$ and $ng$ in place of 1, $g^{-1}$, and $g^n$. Moving forward, we will refer to a group $(G, *)$ simply as $G$ if the operation is clear.

**Example 2.3.** As discussed in the informal section of Definition 2.1, the integers under addition form an abelian group. That is, $(\mathbb{Z}, +)$ is an abelian group.

**Example 2.4.** Any set $G$ with just one element, $G = \{e\}$, is a group by the binary operation $e * e = e$. Indeed, this operation is associative, $e$ is the identity, and $e$ is its own inverse. This group is abelian as well.

We call this the **trivial group**.

**Example 2.5.** The integers under multiplication do not form a group because no elements besides 0, $-1$, and 1 have multiplicative inverses. That is, $(\mathbb{Z}, \cdot)$ is not a group.

**Example 2.6.** The real numbers under addition, $(\mathbb{R}, +)$, forms a group for similar reasons to $(\mathbb{Z}, +)$. However, the real numbers under multiplication, $(\mathbb{R}, \cdot)$, does not form a group because 0 does not have a multiplicative inverse. However, if we define

$$\mathbb{R}^\times = \mathbb{R} \setminus \{0\},$$

then $(\mathbb{R}^\times, \cdot)$ forms an abelian group.

**Example 2.7.** The integers under subtraction, $(\mathbb{Z}, -)$, is not a group. Although $-$ admits an identity and inverses, $-$ is not associative. For example,

$$(1 - 2) - (-3) = 2$$

while

$$1 - (2 - (-3)) = -4.$$

**Example 2.8.** Fix a positive integer $n$. The space $M(n,\mathbb{R})$ of $n \times n$ matrices is not a group under multiplication. Although the multiplication is associative and the diagonal matrix of 1's is an identity, not every matrix admits an inverse. For example,

$$\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} \in M(2,\mathbb{R})$$

has determinant 0, thus is not invertible.

If we restrict to the invertible matrices, $GL(n,\mathbb{R}) \subset M(n,\mathbb{R})$, then $GL(n,\mathbb{R})$ is a group under multiplication. Note $(GL(n,\mathbb{R}),\cdot)$ is not abelian. For example,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

but

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

**Example 2.9** (Dihedral group)**.** Our first unfamiliar example: the symmetries of a triangle, $D_3$, form a group under composition $\circ$.

By symmetries of a triangle, we mean this. Given an equilateral triangle, counterclockwise rotations by 0, $2\pi/3$, and $4\pi/3$ about the center result in the same triangle with the vertices in different positions. We call these rotations $e$, $R$, and $R^2$ respectively. In Figure 1, these rotations correspond to (a), (b), and (c).

Another three symmetries are given by reflections across the three altitudes of the triangle. We call these reflections $F_1$, $F_2$, and $F_3$, pictured in (d), (e), and (f) of Figure 1. These symmetries comprise the elements of $D_3$, making six in total.

The operation is composition — that is, if $A, B \in D_3$, then $A \circ B$ means apply $B$ to the triangle, then apply $A$ to the resulting triangle (see Examples in Figure 2). Note that $R^2$ is simply $R \circ R$, justifying the notation. We can work out by force that for any two elements $A, B \in D_3$, $A \circ B \in D_3$, so $\circ$ is a well-defined binary operation $D_3 \times D_3 \to D_3$.

In fact, $\circ$ is associative, $e$ is the identity, and every element has an inverse ($R^{-1} = R^2$, the remaining elements are their own inverses). It follows that $(D_3, \circ)$ is a group. Note that $D_3$ is not abelian by the Example in Figure 2.

In general, the symmetries of an $n$-gon is denoted $D_n$, and $(D_n, \circ)$ is a group with $2n$ elements. This is known as the **Dihedral group**.

**Example 2.10** (Symmetric group)**.** Let $n \in \mathbb{Z}$ be positive. A **permutation** on the set $S = \{1, 2, \ldots, n\}$ is a bijection $S \to S$. We denote by $S_n$ the set of all permutations of $S$. Note that the identity map $\mathbb{1} \in S_n$ that sends every element to itself is an identity under composition. That is, for every $\sigma \in S_n$,

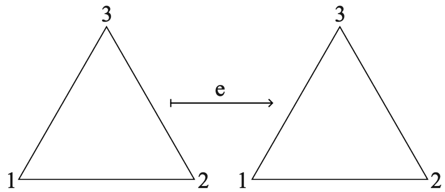
$$\sigma \circ \mathbb{1} = \mathbb{1} \circ \sigma = \sigma.$$

Moreover, bijections are invertible, so $\sigma \in S_n$ implies there exists $\sigma^{-1} \in S_n$. Finally, function composition is associative, so $(S_n, \circ)$ is a group. We call this the **symmetric group on $n$ indices**.

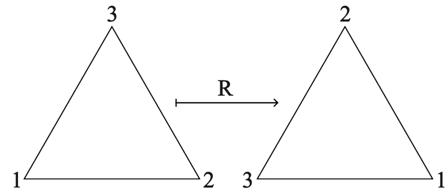
**Definition 2.11.**

**Informal.** Note that the Dihedral group $D_3$ of Example 2.9 is kind of similar to the symmtric group $S_3$ of Example 2.10. For example, note that the rotation $R$ by $2\pi/3$ of Figure 1(b) maps the indices like

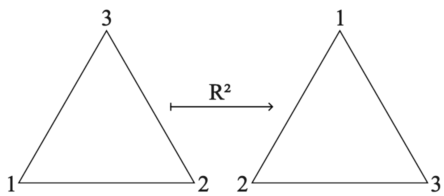
$$1 \mapsto 3,\ 2 \mapsto 1,\ 3 \mapsto 2.$$



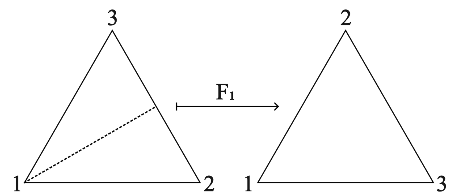

(A) Rotation by 0.



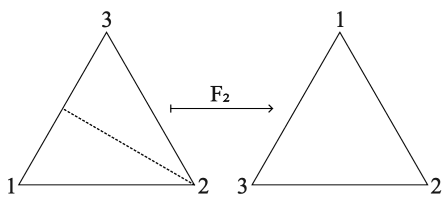

(B) Rotation counterclockwise by $2\pi/3$



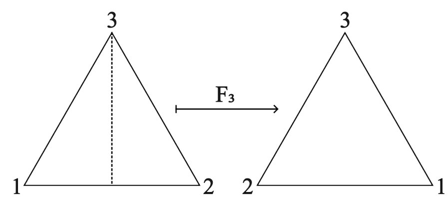

(C) Rotation counterclockwise by $4\pi/3$.




(D) Reflection over altitude from bottom left vertex.

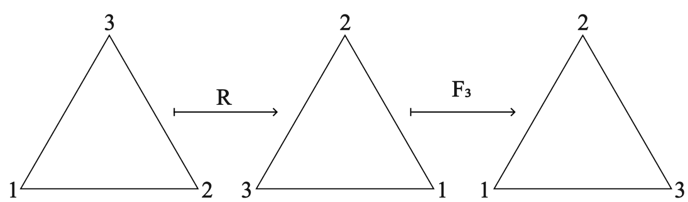



(E) Reflection over altitude from bottom right vertex.

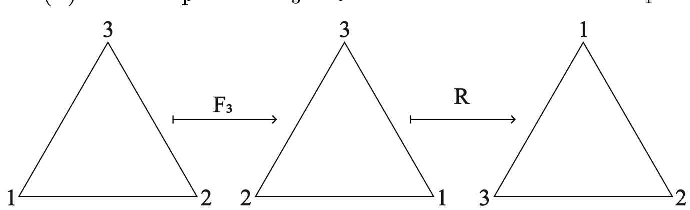



(F) Reflection over altitude from top vertex.

FIGURE 1. Symmetries of the triangle.




(A) The composition $F_3 \circ R$. Note this is the same as $F_1$.




(B) The composition $R \circ F_3$. Note this is the same as $F_2$.

FIGURE 2. Examples of compositions in $D_3$. Observe $F_3 \circ R \neq R \circ F_3$.

But this is a permutation of $\{1,2,3\}$, we can define a function $D_3 \to S_3$ that sends $R$ to the permutation above and the other symmetries to their corresonding permutations. This hints at some deep similarity between $D_3$ and $S_3$ — in fact, we say they are *isomorphic*, denoted $D_3 \cong S_3$.

There are also weaker relationships between groups. For example, it is true that for any $A, B \in GL(n, \mathbb{R})$ (see Example 2.8),

$$\det(AB) = \det(A)\det(B).$$

That is, the determinant turns multiplication in $GL(n,\mathbb{R})$ into multiplication in $\mathbb{R}^\times$ (see Example 2.6). So somehow, the determinant is telling us the multiplications on $GL(n,\mathbb{R})$ and $\mathbb{R}^\times$ induce somewhat similar group structures. We say that det is a *homomorphism*. An isomorphism is a homomorphism that is really nice — one that is invertible.

**Formal.** Let $(G, *)$, $(G', \star)$ be groups. A **group homomorphism** from $G$ to $G'$ is a map

$$\varphi : G \to G'$$

such that for all $a, b \in G$,

$$\varphi(a * b) = \varphi(a) \star \varphi(b).$$

A group homomorphism $\varphi : G \to G'$ is a **group isomorphism** if $\varphi$ is invertible. Then we denote $G \cong G'$.

**Example 2.12.** Given any group $(G, *)$, the identity map $\mathrm{id} : G \to G$, defined by

$$\mathrm{id}(g) = g$$

for all $g \in G$, is a group homomorphism. To see this, observe that for any $a, b \in G$,

$$\mathrm{id}(a * b) = a * b = \mathrm{id}(a) * \mathrm{id}(b).$$

In fact, id is also its own inverse, thus an isomorphism.

A concrete example of this is the familiar function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$.

**Example 2.13.** Given any groups $(G, *)$, $(G', \star)$, where we denote the identity of $G'$ by $\mathbb{1}'$, the function $\varphi : G \to G'$ defined by

$$\varphi(g) = \mathbb{1}'$$

for all $g \in G$ is a group homomorphism.

**Example 2.14.** As we discussed in Definition 2.11, the determinant

$$\det : GL(n, \mathbb{R}) \to \mathbb{R}^\times$$

is a group homomorphism. In fact, one can show det is a surjective but not injective homomorphism.

**Example 2.15.** The function $\varphi : \mathbb{Z} \to \mathbb{Z}$ that maps

$$\varphi(a) = 2a$$

for all $a \in \mathbb{Z}$ is a group homomorphism. To see this, observe that for any $a, b \in \mathbb{Z}$,

$$\varphi(a + b) = 2(a + b) = 2a + 2b = \varphi(a) + \varphi(b).$$

Additionally, if $a \neq b$, then $2a \neq 2b$, so $\varphi$ is injective. However, $\varphi$ is not surjective as it does not reach any odd numbers. Note though, that $\varphi$ is bijective as a function from $\mathbb{Z}$ to $2\mathbb{Z}$. We will see another example of this occuring below.

**Example 2.16.** A very interesting example is the exponential map $\exp : \mathbb{R} \mapsto \mathbb{R}^\times$ given by

$$\exp(x) = e^x$$

for all $x \in \mathbb{R}$. Note that this is a homomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}^\times, \cdot)$ because for all $x, y \in \mathbb{R}$,

$$\exp(x + y) = e^{x+y} = e^x e^y = \exp(x) \cdot \exp(y).$$

Note that $e^x$ is an invertible function from $\mathbb{R}$ to the positive real numbers $\mathbb{R}^+$ (since the inverse is $\log(x)$), so exp is injective, but only surjective on $\mathbb{R}^+$. Ultimately, exp is not surjective on all of $\mathbb{R}$.

This leads in to our next topic.

**Definition 2.17** (Subgroups [DF04, p. 22])**.**

**Informal.** The homomorphisms examined in Examples 2.15 and 2.16 were almost isomorphisms. For $\varphi$ in Example 2.15, $\varphi$ was bijective as a function $\mathbb{Z} \to 2\mathbb{Z}$. For exp in 2.16, exp was bijective as a function $\mathbb{R} \to \mathbb{R}^+$. In fact, we can show $(2\mathbb{Z}, +)$ and $(\mathbb{R}^+, \times)$ are groups in their own right, so $\varphi$ and exp give isomorphisms $\mathbb{Z} \cong 2\mathbb{Z}$ and $(\mathbb{R}, +) \cong (\mathbb{R}^+, \cdot)$ respectively. They give isomorphisms to groups that lie inside other groups.

This sort of situation happens quite frequently: within a group are often other groups, which are in fact groups by the same operation. If a group $G$ contains a group $H$, we call $H$ a *subgroup* of $G$.

**Formal.** Let $(G, *)$ be a group, and let $H$ be a nonempty subset of $G$. If we have the properties:

(i) **Closure under inverses**. For all $h \in H$, $h^{-1} \in H$.
(ii) **Closure under** $*$. For all $h, k \in H$, $h * k \in H$,

then we say $(H, *)$ is a **subgroup** of $G$. We often denote this by $H \leq G$.

Note these conditions tell us $(H, *)$ is a group in its own right, independent of $G$.

**Example 2.18.** Let $G$ be any group. Then $\{1\}, G \leq G$ are easy examples of subgroups.

**Example 2.19.** As noted in Definition 2.17, $2\mathbb{Z} \leq \mathbb{Z}$ and $\mathbb{R}^+ \leq \mathbb{R}^\times$.

**Example 2.20.** The negative real numbers $\mathbb{R}^-$ not a subgroup of $\mathbb{R}^\times$ because $-1 \in \mathbb{R}^-$, but $(-1)(-1) = 1$ is not in $\mathbb{R}^\times$. Therefore, $\mathbb{R}^-$ is not closed under multiplication.

**Example 2.21.** Recall the definition of $GL(n, \mathbb{R})$ from Example 2.8. Let

$$SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : \det(A) = 1\}.$$

This is a subgroup by some properties of matrices:

(i) for any $A \in SL(n, \mathbb{R})$, $\det(A^{-1}) = 1$ implies $A^{-1} \in SL(n, \mathbb{R})$,
(ii) for any $A, B \in SL(n, \mathbb{R})$, $\det(AB) = 1$ implies $AB \in SL(n, \mathbb{R})$.

**Example 2.22.** From Example 2.9, note that $\{e, R_1, R_2\} \leq D_3$. Indeed, the inverses of rotations are all rotations, and the compositions of rotations are rotations as well.

We also have $\{e, F_i\} \leq D_3$ for all $i = 1, 2, 3$. This is because $F_i$ is always its own inverse.

From the discussion in Definition 2.17, we have managed to recover isomorphisms from the homomorphisms in Examples 2.15 and 2.16 by considering subgroups. But can we do the same for a homomorphism like det in Example 2.14, which is surjective but not injective? We would like to "shrink" the domain in such a way that no two elements take on the same value, while still maintaining a group structure. To do something like this, we will need to take a detour to set theory.

**Definition 2.23** (Partitions [DF04, p. 3])**.**

**Informal.** If we're working with some set $A$, it's often useful to separate the elements of $A$ into smaller subsets, maybe based on some property. For example, $\mathbb{Z}$ can be divided up into the subsets

$$\begin{aligned} 3\mathbb{Z} &= \{\ldots, -6, -3, 0, 3, 6, \ldots\}, \\ 3\mathbb{Z}+1 &= \{\ldots, -5, -2, 1, 4, 7, \ldots\}, \\ 3\mathbb{Z}+2 &= \{\ldots, -4, -1, 2, 5, 8, \ldots\}. \end{aligned}$$

These subsets divide up $\mathbb{Z}$ really nicely in the sense that none of their elements overlap and their union is all of $\mathbb{Z}$. When we divide up a set $A$ into subsets whose intersections with each other are empty and union is the whole set, we get a *partition* of $A$.

**Formal.** A **partition** of a nonempty set $A$ is a collection $\{A_i : i \in I\}$ of nonempty subsets of $A$ (here, $I$ is an **indexing set** that helps us keep track of the sets in the collection. Common ones are the positive integers $\mathbb{Z}^+$ and $\mathbb{R}$) such that

$$A = \bigcup_{i \in I} A_i$$

is the union of all the $A_i$ and

$$A_i \cap A_j = \varnothing$$

for all $i, j \in I$ such that $i \neq j$.

**Example 2.24.** Let $A$ be a nonempty set. The partition consisting of just the subset $A \subset A$ is technically a partition of $A$.

**Example 2.25.** Let $A$ be a nonempty set. The partition consisting of the one-element set $\{a\} \subset A$ for all $a \in A$ is a partition on $A$.

**Example 2.26.** From the discussion in Definition 2.23, $\mathbb{Z}$ is partitioned by the subsets $3\mathbb{Z}$, $3\mathbb{Z}+1$, and $3\mathbb{Z}+2$.

These partitions are intimately related to the next idea.

**Definition 2.27** (Equivalence relations [DF04, p. 3])**.**

**Informal.** Seemingly a departure from what we've been talking about, we would now like to generalize the idea of equality. Note that equality, $=$, has the following properties in a set $A$:

(i) for all $a \in A$, $a = a$,
(ii) for all $a, b \in A$, $a = b$ implies $b = a$,
(iii) for all $a, b, c \in A$, $a = b$ and $b = c$ implies $a = c$.

We call any relationship of elements in a set satisfying the same types of properties is an *equivalence relation*.

**Formal.** A **relation** on a nonempty set $A$ is a subset $R$ of $A \times A$, and we write $a \sim b$ if and only if $(a, b) \in R$. An **equivalence relation** on $A$ is a relation on $A$ satisfying:

(i) **Reflexivity**. For all $a \in A$, $a \sim a$.
(ii) **Symmetry**. For all $a, b \in A$, $a \sim b$ implies $b \sim a$.
(iii) **Transitivity**. For all $a, b, c \in A$, $a \sim b$ and $b \sim c$ implies $a \sim c$.

**Example 2.28.** An easy equivalence relation on a nonempty set $A$ is the entire set $R = A \times A$. This amounts to saying that for all $a, b \in A$, $a \sim b$.

**Example 2.29.** The familiar $=$ relation is given by the subset
$$R = \{(a, a) \in A \times A : a \in A\}.$$
That is, every element is related only to itself.

**Example 2.30.** $\cong$ is an equivalence relation on any set of groups. If $G$ is any group, then $\mathrm{id} : G \to G$ (see Example 2.12) is an isomorphism, so $G \cong G$. Symmetry comes from the fact inverses of bijective group homomorphisms are themselves group homomorphisms. Transitivity comes from the fact compositions of homomorphisms are still group homomorphisms. We will omit the proofs of these.

**Example 2.31.** $\leq$ is not an equivalence relation on $\mathbb{Z}$. Although $\leq$ is reflexive and transitive, $\leq$ fails symmetry. For example, $4 \leq 5$, but $5 \not\leq 4$.

**Example 2.32.** Define a relation on $\mathbb{Z}$ via $a \sim b$ if and only if $a$ has the same remainder as $b$ when divided by 3. This is an equivalence relation:

(i) For all $a \in \mathbb{Z}$, $a$ has the same remainder mod 3 as itself.
(ii) For all $a, b \in \mathbb{Z}$, if $a$ has the same remainder as $b$ mod 3, then $b$ has the same remainder as $a$ mod 3.
(iii) For all $a, b, c \in \mathbb{Z}$, $a \sim b$ and $b \sim c$ means $a$ has the same remainder as $b$ when divided by 3, but $c$ also has the same remainder as $b$ when divided by 3. Thus, $a$ must have the same remainder as $c$ when divided by 3, hence $a \sim c$.

**Definition 2.33** ([DF04, p. 3])**.**

**Informal.** Given an equivalence relation $\sim$ on a set $A$ and an element $a \in A$, we can think about the subset of elements that relate to $A$. This is called the *equivalence class* of $a$ with respect to $\sim$. We denote this by $[a]_\sim$.

**Formal.** Let $A$ be a nonempty set, $\sim$ an equivalence relation on $A$. Then for all $a \in A$, the **equivalence class of** $a$ with respect to $\sim$ is defined as
$$[a]_\sim = \{b \in A : a \sim b\}.$$

**Example 2.34.** In Example 2.28, the equivalence class of any element is the whole set. That is, for all $a \in A$,
$$[a]_\sim = A.$$
Note this gives the sets for the partition for 2.24.

**Example 2.35.** In Example 2.29, the equivalence class of any element is the set containing just itself. That is, for all $a \in A$,
$$[a]_\sim = \{a\}.$$
Note this gives the sets in the partition for Example 2.25.

**Example 2.36.** In Example 2.32, the equivalence class of any element is one of $3\mathbb{Z}$, $1+3\mathbb{Z}$, and $2+3\mathbb{Z}$ depending on if the remainder of that element mod 3 is 0, 1, or 2 respectively. Note that the equivalence classes give precisely the subsets for the partition in 2.26.

In Examples 2.34, 2.35, and 2.36, we see that equivalence relations induce partitions via equivalence classes. Interestingly, this pattern always holds, and in fact goes both ways.

**Theorem 2.37** ([DF04, p. 3])**.** *Let $A$ be a nonempty set.*

*(i) If $\sim$ is an equivalence relation on $A$, then the set of equivalence classes $\sim$ forms a partition of $A$.*

*(ii) Conversely, if $\{A_i : i \in I\}$ is a partition of $A$, then there is an equivalence relation on $A$ whose equivalence classes give that partition.*

*Proof.* (i): Let $A$ be a nonempty set, and suppose $\sim$ is an equivalence relation on $A$. Then we will show set of equivalence classes of $A$ with respect to $\sim$ form a partition of $A$.

(a) First,
$$A = \bigcup_{a \in A} [a]_\sim$$
because for all $a \in A$, $a \in [a]_\sim$.

(b) Second, if $[a]_\sim = [b]_\sim$ for some $a, b \in A$, then $a \in [a]_\sim$ implies $a \in [b]_\sim$. But this means $a \sim b$. We can then show this means $[a]_\sim$ and $[b]_\sim$ are actually the same set. Therefore, equivalence classes that are different from each other must have empty intersection, lest they actually be equal.

(ii): Let $A$ be a nonempty set, and suppose $\{A_i : i \in I\}$ is a partition of $A$. Then define the relation $\sim$ on $A$ where for all $a, b \in A$, $a \sim b$ if and only if $a, b \in A_i$ for some $i \in I$. We will show this is an equivalence relation.

(a) For all $a \in A$, by the definition of a partition of $A$, $a \in A_i$ for some $i \in I$, so $a \sim a$.
(b) For all $a, b \in A$, $a, b \in A_i$ implies $b, a \in A_i$ pretty self-evidently. Thus, $a \sim b$ implies $b \sim a$.
(c) For all $a, b, c \in A$, $a, b \in A_i$ and $b, c \in A_j$ for some $i, j \in I$ implies $b \in A_i \cap A_j$. Thus, $A_i \cap A_j \neq \varnothing$, which means $i = j$, so $a \sim c$.

Now we will show the equivalences classes of $\sim$ give the desired partition. Given any $i \in I$, we have
$$A_i = [a]_\sim$$
for any $a \in A_i$ by definition of $\sim$. Likewise, for any $a \in A$, $a \in A_i$ for some $i \in I$, so
$$[a]_\sim = A_i.$$
We conclude the equivalence classes of $\sim$ give the desired partition $\{A_i : i \in I\}$. □

So partitions and equivalence relations are really equivalent ideas. We will now circle back to groups. The strategy will be to "shrink down" a group by partitioning that group, then turning the set of subsets in that partition into a new group. First, we will need the following more technical definitions.

**Definition 2.38** (Normal subgroup [DF04, p. 82]). Let $G$ be a group, $N \leq G$ a subgroup of $G$. We call $N$ a **normal subgroup** of $G$ if for all $n \in N, g \in G$, $gng^{-1} \in N$. We denote this $N \trianglelefteq G$.

**Example 2.39.** $3\mathbb{Z} \leq \mathbb{Z}$ is a normal subgroup because given any $a \in 3\mathbb{Z}$, $b \in \mathbb{Z}$,
$$b + a + (-b) = a \in 3\mathbb{Z}.$$
In fact, the same method shows that subgroups of abelian groups are normal.

**Example 2.40.** Recall the subgroup $\{e, F_1\} \leq D_3$ in Example 2.22. This is not a normal subgroup because
$$R_2 \circ F_1 \circ R_1 = F_3 \notin \{e, F_1\}.$$

**Example 2.41.** Recall from Example 2.21 that $SL(n,\mathbb{R})$ is a subgroup of $GL(n,\mathbb{R})$. In fact, $SL(n,\mathbb{R})$ is a normal subgroup of $GL(n,\mathbb{R})$. To see this, let $A \in SL(n,\mathbb{R}), B \in GL_n(\mathbb{R})$ be arbitrary. Then

$$\det\left(BAB^{-1}\right) = \det(B)\det(A)\det(B)^{-1} = \det(A) = 1,$$

so $A \in SL(n,\mathbb{R})$. The subset

$$SL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) : \det(A) = 1\}$$

is a normal subgroup.

**Definition 2.42** (Cosets [DF04, p. 77])**.** Let $(G,*)$ be a group, $H \leq G$ a subgroup of $G$. For any $g \in G$, define the set

$$gH = \{g * h \in G : h \in H\}.$$

We call these left **cosets** of $H$ in $G$. Any element of a coset is a called a **representative** for the coset.

If $(G,+)$ is abelian, we often write $g+H$ in place of $gH$.

**Example 2.43.** Let $G$ be a group. Recall that $\{1\}, G \leq G$ are subgroups. In fact, $\{1\}, G \trianglelefteq G$ are normal subgroups as well.

**Example 2.44.** The sets $3\mathbb{Z}$, $1+3\mathbb{Z}$, and $2+3\mathbb{Z}$ defined in Definition 2.23 are cosets of the subgroup $3\mathbb{Z} \leq \mathbb{Z}$. Numbers like $1, -2, 10$ are all representatives of $1+3\mathbb{Z}$. Note the name representative makes sense, since

$$1+3\mathbb{Z} = -2+3\mathbb{Z} = 10+3\mathbb{Z}.$$

**Example 2.45.** The cosets of $SL(n,\mathbb{R})$ (see Example 2.21) are precisely the sets of elements with the same determinant. That is, suppose $A \in GL(n,\mathbb{R})$ has determinant $u$ for some $u \in \mathbb{R}$. Then every element of $A(SL(n,\mathbb{R}))$ has determinant $u$. Conversely, if another matrix $B \in GL(n)$ has determinant $n$, then

$$B = A(A^{-1}B)$$

where $\det\left(A^{-1}B\right) = 1$ implies $B \in A(SL(n,\mathbb{R}))$. Therefore, we can say det is injective on the cosets of $SL(n,\mathbb{R})$. Then to salvage an isomorphism out of det, we need only create a new group whose elements are the cosets of $SL(n,\mathbb{R})$. We will show how to do this.

**Proposition 2.46.** *Let $G$ be a group, $H \leq G$ any subgroup. Then the cosets of $H$ in $G$ partition $G$.*

*Proof.* Let $\sim$ be the equivalence relation on $G$ defined by $a \sim b$ if and only if $a$ and $b$ are in the same coset of $H$ for all $a, b \in G$. This is an equivalence relation because

(i) for any $a \in G$, $a \in aH$, so $a \sim a$,
(ii) for any $a, b \in G$, $a, b \in gH$ implies $b, a \in gH$, so $b, a \in G$.
(iii) for any $a, b, c \in G$, $a, b \in gH$ and $b, c \in g'H$ implies there exist $h, h_1', h_2' \in H$ such that

$$b = gh,\ b = g'h_1',\ c = g'h_2'.$$

We deduce using inverses that

$$g' = bh_1'^{-1} = ghh_1',$$

so

$$c = ghh_1'h_2' \in gH.$$

Thus, $a \sim c$.

Since $\sim$ is an equivalence relation, the equivalence classes of $\sim$ partition $G$. Note these equivalence classes are precisely the cosets of $H$. $\square$

**Proposition 2.47.** *Let $G$ be a group, $N \trianglelefteq G$ a normal subgroup. Then for any two cosets $aN, bN$, the coset $abN$ is independent of the choice of representatives. That is, given any other representatives $a', b'$ of $aN$ and $bN$ respectively, $a'b'N = abN$.*

*Proof.* Let $g \in abN$. Then we can write $g = abn$ for some $n \in N$. If $a' \in aN$, $b' \in bN$, then we can also write $a' = an_1$, $b' = bn_2$ for $n_1, n_2 \in N$. Then

$$g = abn = (a'n_1^{-1})(b'n_2^{-1})n = a'(b'b'^{-1})n_1^{-1}b'n_2^{-1}n = a'b'(b'^{-1}n_1^{-1}b')n_2^{-1}n \in a'b'N$$

because $b'^{-1}n_1b' \in N$ by $N$ being normal. By the same argument the other direction, $abN = a'b'N$. $\square$

**Definition 2.48** (Quotient groups)**.**

**Informal.** By putting a partition $\{A_i : i \in I\}$ on a group $G$, we can form a new group $\overline{G}$ whose elements are the subsets $A_i$ themselves. A concrete example is helpful. Recall the partition of 2.36 on $\mathbb{Z}$ by the cosets $3\mathbb{Z}$, $1 + 3\mathbb{Z}$, and $2 + 3\mathbb{Z}$. The elements of our new group, which we denote $\mathbb{Z}/3\mathbb{Z}$, are as follows:

$$\mathbb{Z}/3\mathbb{Z} = \{3\mathbb{Z}, 1 + 3\mathbb{Z}, 2 + 3\mathbb{Z}\}.$$

We emphasize that the sets themselves have become elements. $\mathbb{Z}/3\mathbb{Z}$ is a group in the following way. If we define the sum of cosets of $3\mathbb{Z}$ by

$$(a + 3\mathbb{Z}) + (b + 3\mathbb{Z}) = (a + b) + 3\mathbb{Z},$$

then

$$3\mathbb{Z} + 3\mathbb{Z} = 3\mathbb{Z},\ 3\mathbb{Z} + (1 + 3\mathbb{Z}) = 1 + 3\mathbb{Z},\ (1 + 3\mathbb{Z}) + (2 + 3\mathbb{Z}) = 3\mathbb{Z},\ \text{etc.}$$

Note the "niceness" here. The sums of the sets in $\mathbb{Z}/3\mathbb{Z}$ always give another element in $\mathbb{Z}/3\mathbb{Z}$. Moreover, the result stays the same regardless of which representative we choose for each coset. All in all, we get a group structure on $\mathbb{Z}/3\mathbb{Z}$, which we call a the *quotient group* of $\mathbb{Z}$ with respect to the subgroup $3\mathbb{Z}$. This only works because $3\mathbb{Z}$ is normal.

**Formal.** Let $(G, *)$ be a group, $N \trianglelefteq G$ be a normal subgroup of $G$. Then define $G/N$ to be the set of cosets of $N$ in $G$. That is,

$$G/N = \{gN : g \in G\}.$$

The operation is as follows: for all $a, b \in G$,

$$(aN) * (bN) = (a * b)N.$$

By Propositions 2.46 and 2.47, the definition of this operation is unambiguous — every element uniquely represents a coset and the operation is independent of which representative we choose. We can show $(G/N, *)$ forms a group, which we call the **quotient group** of $G$ with respect to $N$.

**Example 2.49.** We needed $N$ to be normal in Definition 2.48 because otherwise, the group operation is not well-defined. For instance, we showed in Example 2.40 that $\{e, F_1\} \leq D_3$ is not a normal. Call this subgroup $\Delta$. Then

$$R_2\Delta = F_2\Delta = \{R_2, F_2\},$$

but then

$$R_2\Delta \circ (R_2\Delta) = R_1\Delta = \{1, F_3\},$$

while

$$F_2\Delta \circ F_2\Delta = \Delta = \{1, F_1\}.$$

Therefore, the operation is not independent of the representative we choose for the cosets, and so does not have a well-defined output. This is rectified by our choice of $N$ to be normal.

**Remark 2.50.** In fact, if $(G, *)$ is a group with subgroup $H \leq G$, then $(G/H, *)$ is a group if and only if $H$ is normal in $G$.

**Example 2.51.** Recall $\{1\}, G \trianglelefteq G$ are normal subgroups. Then $G/G$ has just one element, while $G/\{1\}$ is isomorphic to $G$. That is, $G/G \cong \{1\}$, $G/\{1\} \cong G$.

**Example 2.52.** By taking the determinant homomorphism (Example 2.14) on the quotient $GL(n, \mathbb{R})/SL(n, \mathbb{R})$ (Example 2.41), we get $\overline{\det} : GL(n, \mathbb{R})/SL(n, \mathbb{R}) \to \mathbb{R}^\times$ via

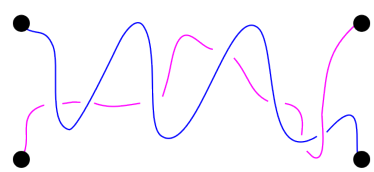
$$\overline{\det}(A(SL_n(\mathbb{R}))) = \det(A)$$

for all $A(SL_n(\mathbb{R})) \in GL(n, \mathbb{R})/SL(n, \mathbb{R})$. We can check that $\overline{\det}$ is independent of the choice of representative, and in fact, is a group isomorphism. Thus, $GL(n, \mathbb{R})/SL(n, \mathbb{R}) \cong \mathbb{R}^\times$.

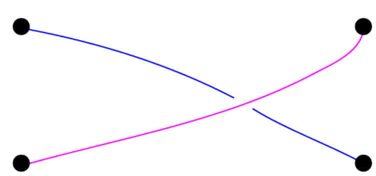
**Example 2.53.** $\mathbb{Z}/3\mathbb{Z}$, as discussed in Definition 2.48, is a quotient group. This group represents arithmetic modulo 3, where we only distinguish elements up to their remainder when divided by 3. This idea can be generalized to $\mathbb{Z}/n\mathbb{Z}$ for any $n \in \mathbb{Z}$ a positive integer.

## 3. Homotopy

Before getting to braids, there is one more thing we need — that is the notion of a "continuous deformation." For example, if we are to talk about braids, there must be a bit of wiggle room in what makes a braid. To illustrate, these messy strands



can easily be turned into the braid



just by nudging the blue strand up a bit and nudging the pink strand down a bit. Maybe it would be a bit unfair or unproductive to say these are different braids, simply because their strands do not strictly occupy the same coordinates in space. But to speak about this precisely, we must make precise what we mean by "nudging."

**Definition 3.1** (Homotopy).

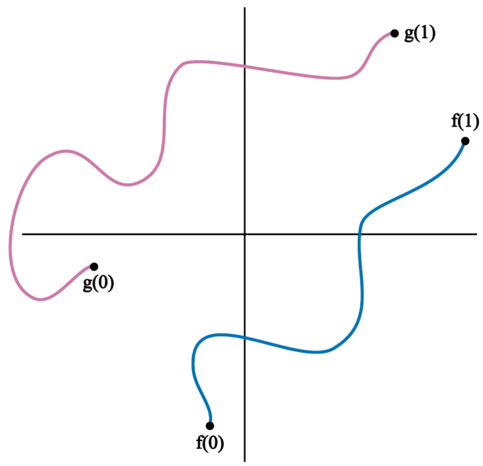



FIGURE 3. Continuous functions $f, g : [0, 1] \to \mathbb{R}^2$

**Informal.** The most basic form of a "nudge" is a *homotopy*. To illustrate, suppose $f$ and $g$ are functions that parameterize a line segment in $\mathbb{R}^2$. So $f, g$ can be a continuous functions $[0, 1] \to \mathbb{R}^2$ as below in Figure 3:

We call these **paths** in $\mathbb{R}^2$. Now let's say we want to "nudge" the path $f$ to be the path $g$. Intuitively, this is something we can do, say if $f$ and $g$ represent strings on a table. To express this mathematically, we must write a function that parameterizes the paths themselves.

At time 0, the function must output the path $f$, and at time 1, the function must output the path $g$. All this must be done continuously, without skipping any space or breaking the strings, as below in Figure 4:

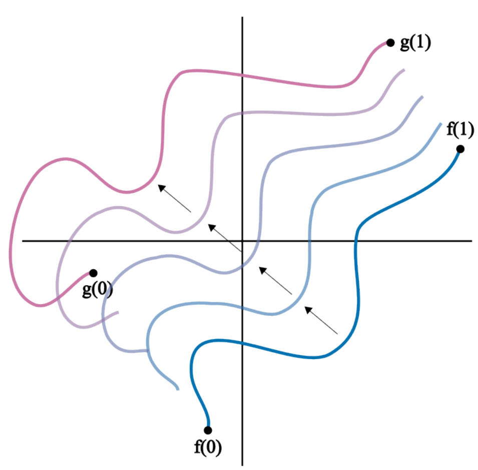



FIGURE 4. A homotopy from $f$ to $g$.

Perhaps this is easier to do than it sounds. In this case, the function is given by $F : [0,1]^2 \to \mathbb{R}^2$ where for all $(s,t) \in [0,1]^2$,

$$F(s,t) = f(s)(1-t) + g(s)t.$$

Observe that at time $t = 0$, $F(s,0)$ gives the path $f(s)$, and at time $t = 1$, $F(s,1)$ is the desired path $g(s)$. As $F$ is a sum of products of continuous functions, $F$ is itself continuous, so no breaking or skipping occurs.

**Formal.** Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be arbitrary subsets, and $f, g : U \to V$ continous functions. A **homotopy** from $f$ to $g$ is a continuous function

$$F : U \times [0,1] \to V$$

such that

$$F(x,0) = f(x),$$
$$F(x,1) = g(x).$$

When such a homotopy exists, we say $f$ and $g$ are **homotopic**.

To simply notation, we often denote $F(x,t) = F_t(x)$ for all $t \in [0,1]$, where $F_t$ is a function $U \to V$.

Moving forward, we assume all functions are continuous.

**Proposition 3.2.** *Let $\sim$ denote the relation on functions $U \to V$ where $f \sim g$ if and only if there exists a homotopy from $f$ to $g$. Then $\sim$ is an equivalence relation (Def. 2.27).*

*Proof.*

**Informal.** Homotopy behaves quite like equality, on many many levels. But it all starts with homotopy being an equivalence relation. Every function is homotopic to itself simply by a homotopy that keeps it still for all $t \in [0,1]$. Moreover, if $f$ can be deformed to $g$ by a homotopy, then it makes sense we can deform $g$ back to $f$ by reversing the motion. And, if we can deform $f$ to $g$, then $g$ to $h$, we can $f$ deform to $h$ in the same time frame just by performing both deformations twice as fast in succession.

**Formal.** We must prove $\sim$ is reflexive, symmetric, and transitive.

(i) To see $\sim$ is reflexive, let $f : U \to V$ be arbitrary. Then the function

$$F : U \times [0,1] \to V$$

defined by

$$F(x,t) = f(x)$$

is a homotopy from $f$ to itself. Thus $f \sim f$.

(ii) To see $\sim$ is symmetric, suppose we have $f, g : U \to V$ such that $f \sim g$. That is, there is a homotopy

$$F : U \times [0,1] \to V.$$

Then we define

$$\overline{F} : U \times [0,1] \to V$$

by

$$\overline{F}(x,t) = F(x, 1-t).$$

Then

$$\overline{F}(x,0) = F(x,1) = g(x),$$
$$\overline{F}(x,1) = F(x,0) = f(x)$$

shows $\overline{F}$ is indeed a homotopy from $g$ to $f$, so $g \sim f$.

(iii) To see $\sim$ is transitive, suppose $f, g, h : U \to V$ satisfy $f \sim g$, $g \sim h$. Then there exist

$$F, G : U \times [0,1] \to V$$

where $F$ is a homotopy from $f$ to $g$ and $G$ is a homotopy from $g$ to $f$. Then define

$$H : U \times [0,1] \to V$$

via

$$H(x,t) = \begin{cases} F(x,2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ G(x,2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

One can check $H$ is a homotopy from $f$ to $h$, so $f \sim h$.

□

Since $H$ is an equivalence relation, it is unambiguous to say $f$ and $g$ are homotopic if $f \sim g$.

**Remark 3.3.** For readers who know topology, the definition of homotopy extends to any maps $f : X \to Y$. The other definitions in this section will also have natural topological generalizations.

**Example 3.4.** As may be evident from the discussion in Definition 3.1, given any paths $f, g : [0,1] \to \mathbb{R}^2$, there is is a homotopy from $f$ to $g$ given by $F : [0,1]^2 \to \mathbb{R}^2$,

$$F(s,t) = f(s)(1-t) + g(s)t.$$

**Example 3.5.** The existence of homotopies depends on the choice of codomain. Given the paths $f, g : [0,1] \to \mathbb{R}^2 \setminus \{(0,0)\}$ defined by

$$f(t) = (\sin(2\pi t), \cos(2\pi t)),$$
$$g(t) = f(t) + (1,1),$$

pictured in Figure 5 below, there is no function that can move $f$ to $g$ without breaking or skipping at the origin.

**Example 3.6.** Here's a more interesting example. Let $C$ denote the hollow cylinder $S^1 \times [0,1] \subset \mathbb{R}^3$. Let the functions $\mathrm{id} : C \to C$, $r : C \to C$ be defined by

$$\mathrm{id}(x,s) = (x,s),$$
$$r(x,s) = (x,0).$$

They are both continuous. Observe that $r$ is collapsing all of $C$ onto the base circle $S^1 \times \{0\}$. We claim there is a homotopy from id to $r$.

Let $F : C \times [0,1] \to C$ be defined by

$$F((x,s),t) = (x, s(1-t)).$$

This function is continuous and indeed,

$$F((x,s),0) = (x,s) = \mathrm{id}(x,s),$$
$$F((x,s),1) = (x,0).$$

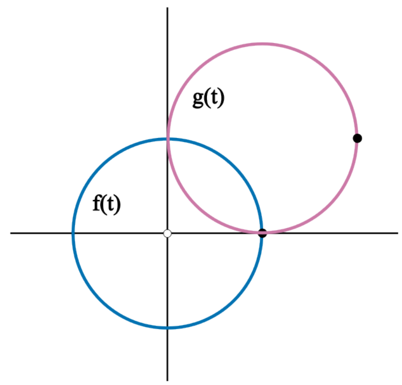



FIGURE 5. $f$ and $g$ are homotopic as paths in $\mathbb{R}^2$, but not as paths in $\mathbb{R}^2 \setminus \{(0,0)\}$. There is no way to move $f$ to $g$ without $f$ at the origin.

This gives $r$ a nice physical interpretation. We can think of id as representing the cylinder $C$ in its original state, not changing the positions of any points. Then, to apply $r$, we squish $C$ down to a circle in accordance to $F$ until we reach $r$. So $r$ can be interpreted squishing $C$ down. This is pictured below in Figure 6.

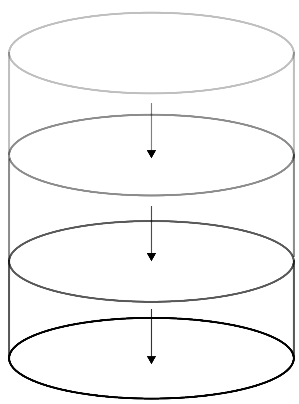


FIGURE 6. Physical interpretation of $r$ from Example 3.6 via the homotopy from id to $r$. As time progresses from $t = 0$ to $t = 1$, we move the points of $C$ in accordance to where they are mapped by $f$ until we get $r$.

We will set up some more terminology.

**Definition 3.7.** Let $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$, and let $f : U \to V$ be a continuous function. If $f$ is a continuous function with a continuous inverse, we say $f$ is a **homeomorphism**. If there is a homeomorphism between $U$ and $V$, we say $U$ and $V$ are **homeomorphic** and write $U \cong V$.

Like homotopy, homeomorphism defines an equivalence relation. Hence, it is unambiguous to say two spaces are homeomorphic.

**Example 3.8.** Let $U \subset \mathbb{R}^n$. The identity map $U \to U$ is continuous and is its own inverse. Hence, it is a homeomorphism. We say $U \cong U$, or $U$ is homeomorphic to itself.

**Example 3.9.** The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is a homeomorphism because $f^{-1} : \mathbb{R} \to \mathbb{R}$, $f^{-1}(x) = x^{1/3}$ is continuous.

**Example 3.10.** The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not a homeomorphism because $f$ is not injective, hence has no inverse $\mathbb{R} \to \mathbb{R}$.

**Example 3.11.** It is not true in general that if a function is continuous and invertible, that its inverse is continuous. For example define $f : [0, 1) \to S^1$ via

$$f(x) = (\cos(2\pi x), \sin(2\pi x))$$

(pictured in Figure 7).

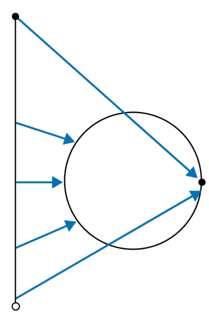


FIGURE 7. $f$ of Example 3.11 is something like this. The interval $[0, 1)$ maps bijectively and continuously onto $S^1$, but reversing this function will break $S^1$.

Observe $f$ wraps the interval $[0, 1)$ into a circle. This is continuous and bijective, hence invertible. However, the inverse requires the circle to taken to the interval $[0, 1)$. This breaks the circle, hence is not continuous.

**Definition 3.12.** Let $f : U \to V$ be a continuous function, and let $W \subset V$ be the image of $f$. If the function $f : U \to W$ given by $f$ is a homeomorphism, we call $f$ an **embedding**.

**Example 3.13.** Embeddings are not as restrictive as homeomorphisms, but give more well-behaved functions than with just continuity. For example, the curve in Figure 8 is continuous, but not an embedding. Likewise, the curves in Figure 5 are not embeddings (though they can be rewritten as embeddings $S^1 \to \mathbb{R}^2$).

Meanwhile, the curves of Figure 3 are embeddings.

This allows us to define a stronger type of homotopy.

**Definition 3.14.** Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$, and $f, g : U \to V$ be embeddings. Then a homotopy $F : U \times I \to V$ from $f$ to $g$ is an **isotopy** if $F(s, t)$ is an embedding for all fixed $t \in [0, 1]$. If there is an isotopy between two embeddings, then we say they are **isotopic**.

In short, an isotopy is a homotopy that is particularly nice.

Like homotopy and homeomorphisms, isotopy defines an equivalence relation. Hence it is unambiguous to say two embeddings are isotopic.

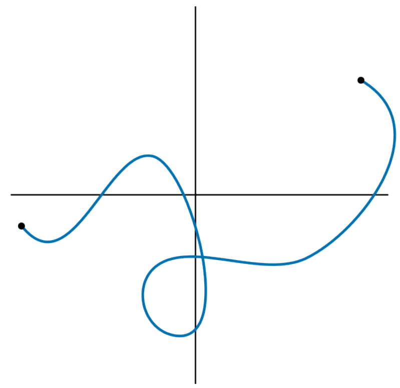


FIGURE 8. Continuous, but not embedded path $[0,1] \to \mathbb{R}^2$.

## 4. Introduction to Braids

We are finally ready to study braids. We will see quickly that this study puts to good use the accumulated knowledge of the previous sections.

**Definition 4.1** (Braid [MK99, p. 3], [FM12, p. 240])**.**

**Informal.** To construct a mathematical braid with $n$ strands, pick $n$ points on a disk in 3-dimensional space. Then pick a parallel disk with the same distinguished points. We obtain a **braid on** $n$ **strands** by drawing non-intersecting lines between the points on the two planes. The lines are not allowed to go backward.

**Formal.** Fix a positive integer $n$. Let $p_1, \ldots, p_n$ be points in the 2-disk $D^2$. A **braid on** $n$ **strands**, or **n**-braid, is a collection of $n$ paths $f_i : [0,1] \to D^2 \times [0,1], 1 \le i \le n$, called **strands**, and a permutation $\overline{f} \in S_n$ such that each of the following holds:

(i) the strands $f_i([0,1])$ are disjoint,
(ii) $f_i(0) = (p_i, 0)$,
(iii) $f_i(1) = (p_{\overline{f}(i)}, 1)$,
(iv) $f_i(t) \in D^2 \times \{t\}$ for all $t \in [0,1]$.

Moving forward, we fix points $p_1, \ldots, p_n$ for all positive integers $n$. We will assume every $n$-braid has these starting/ending points. Moreover, when we say $\beta = \{f_i : 1 \le i \le n\}$ is an $n$-braid, we will also use $\beta$ to mean the subset of strands in $D^2 \times [0,1]$ given by $\beta$. Therefore, if we have a function $f$ whose domain is $D^2 \times [0,1]$, the expression $f(\beta)$ makes sense.

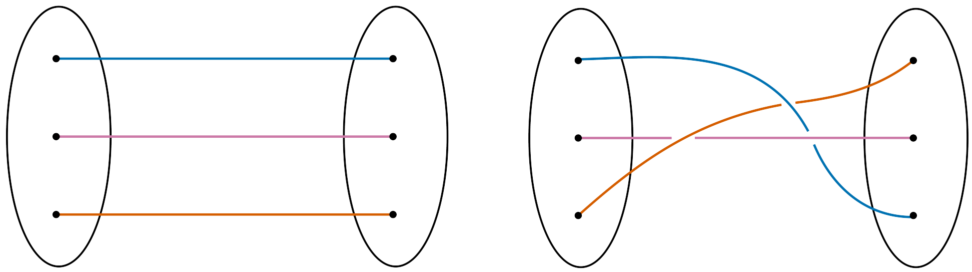
To simplify notation, we will denote $\mathbb{D} = D^2 \times [0,1]$.

**Example 4.2.** In Figure 9, we have two examples of braids on 3 strands, which we refer to as 3-braids. These braids are a subset of the cylinder formed by the parallel disks, or $\mathbb{D}$. Moving forward, we omit the disks in figures.

**Example 4.3.** In Figure 10, we have three non-example of braids.

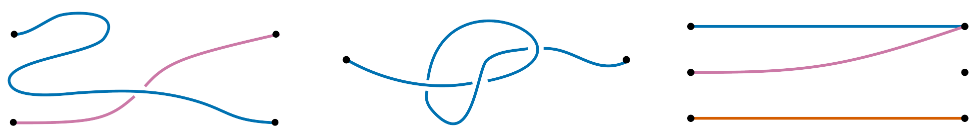
**Definition 4.4** (Braid equivalence [MK99, p. 96])**.**

**Informal.** We want to say two braids are the same if we can pull the strands around so that they are equal. For example, we want to say the three braids in Figure 11 are



(A) (B)

FIGURE 9. (a): On the left, we have what's technically a braid on 3 strands. We can describe this by the paths $f_i(t) = p_i$ for all $t \in [0, 1]$. (b): On the right, a slightly more interesting braid on 3 strands.



(A) (B) (C)

FIGURE 10. (a) and (b) are not braids because they turn back on theselves. (c) is not a braid because the points are not permuted.

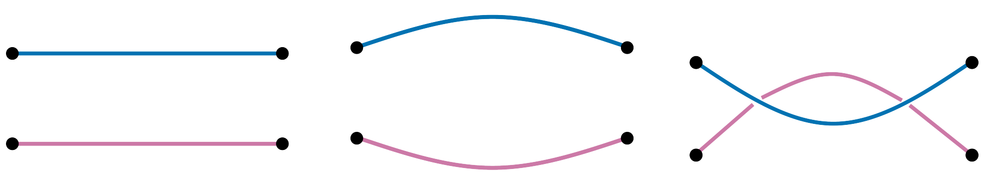


FIGURE 11. These three 2-braids are equivalent.

the same. The rules are, strands cannot cross each other and the starting/ending points cannot change. They are like infinitely elastic rubber bands.

**Formal.** Denote $\mathbb{D} = D^2 \times [0, 1]$. We declare two braids $\beta$ and $\beta'$ **equivalent** if there exists a continuous map

$$h : \mathbb{D} \times [0, 1] \to \mathbb{D}$$

such that:

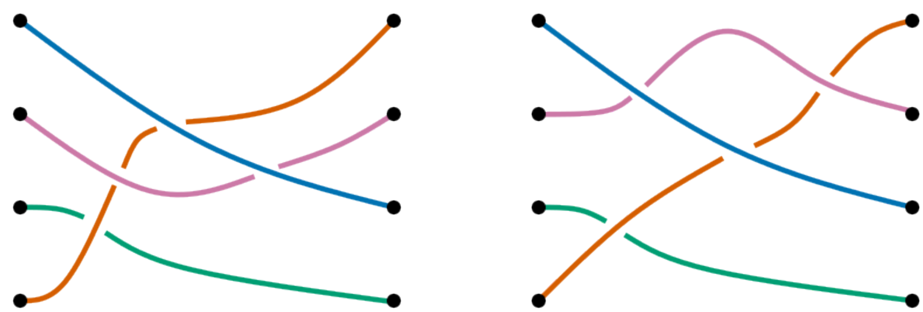
(i) for all $t \in [0, 1]$, $h_t : \mathbb{D} \to \mathbb{D}$ is a homeomorphism,
(ii) for all $t \in [0, 1]$, $h_t$ is the identity on the cylindrical boundary of $\mathbb{D}$. That is, $h_t$ fixes the boundary pointwise.
(iii) $h_0$ is the identity and $h_1(\beta) = \beta'$. We call $h$ an **ambient isotopy**.

An ambient isotopy is even stronger than an isotopy. In the previous definition (Def. 4.4), we obtain an isotopy from $\beta$ to $\beta'$ by hitching a ride on an isotopy of homeomorphisms of the entire space $\mathbb{D}$. It is quite strict, but the ambient isotopy is equivalent to the intuitive notion of pulling and nudging the strands without breaking them or crossing them through each other.

**Author comment.** In truth, I do not know why an ambient isotopy is required as opposed to simply an isotopy of the map $\beta : \{p_1, \dots, p_n\} \times I \to \mathbb{D}$ that fixes the endpoints. Although I understand how all knots can be isotoped to the unknot, I have not seen similar arguments for braids. If anyone has an explanation or counterexample, I would very grateful to hear it!

**Remark 4.5.** There is also a much more easy notion of braid equivalence formulated in terms in elementary moves. It takes no fancy math to understand, and the interested reader can find it in [MK99, p. 4].

**Example 4.6.** The braids in Figure 12 are equivalent.



(A) (B)

FIGURE 12. These 4-braids are equivalent. From (a), shift the pink line upward and nudge the orange line down to get (b).

**Example 4.7.** The braids in Figure 13 are not equivalent.

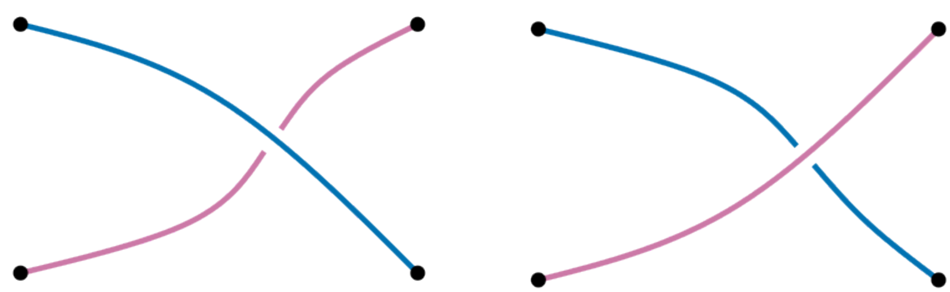


FIGURE 13. These 2-braids are not equivalent

The braid equivalence we have defined turns out to be an equivalence relation (Def. 2.27). But arguably this is what we would expect: after all, we *want* to partition big collections of braids into subsets of braids that are equivalent. As desired, the equivalence class of any braid ends up being the set of all braids equivalent to it.

From now on, when we say braid, we will actually be referring to the whole class of braids we have deemed equivalent. When we say, for example, that $\beta$ is a braid, remember $\beta$ is a representative of some class of equivalent braids.

**Definition 4.8.**

**Informal.** Now that we have put our intuition of equivalent braids into mathematics, we can see this structure is truly a very natural one, despite the complicated formalisms. We can now define a product on the set of braids that is nice enough to form a group structure.

The product of braids $\beta$ and $\beta'$ with the same number of strands is simply connecting $\beta'$ to the end of $\beta$. Technically, braids must be parameterized by the interval $[0, 1]$, so the new braid will have to be reparameterized to travel the two braids twice as fast.

Therefore, we can now form a *braid group*, whose elements are equivalence classes of braids with the some fixed number of strands, and where the multiplication is concatenating the braids.

**Formal.** We define the **braid group on $n$ strands** $B_n$ to be the set of equivalence classes of braids where the product of any two braids $\{f_i : 1 \leq i \leq n\}$, $\{g_i : 1 \leq i \leq n\}$ is $\{h_i : 1 \leq i \leq n\}$ where for all $1 \leq i \leq n$,

$$h_i(t) = f_i(t) * g_i(t) = \begin{cases} f_i(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ g_{\overline{f}(i)}(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Indeed, we can verify the braid product $*$ is well-defined. That is, if we have $\beta_i$ and $\beta_i'$ are equivalent for $i = 1, 2$, then $\beta_1 * \beta_2$ and $\beta_1' * \beta_2'$ are equivalent: there is no ambiguity in choice of representatives. Moreover, we can verify $*$ is associative, that the braid given by $f_i(t) = p_i$ for all $1 \leq i \leq n$ gives an identity element in $B_n$, and that the mirror image of every braid is its inverse.

Every one of these properties can be seen by just drawing pictures. In the process, we would also realize that the notion of equivalence was essential here.

**Example 4.9.** In Figure 14, we see two examples of braid products. Observe in (a) that when we take the product of a braid with the identity, we can pull the braid back to the

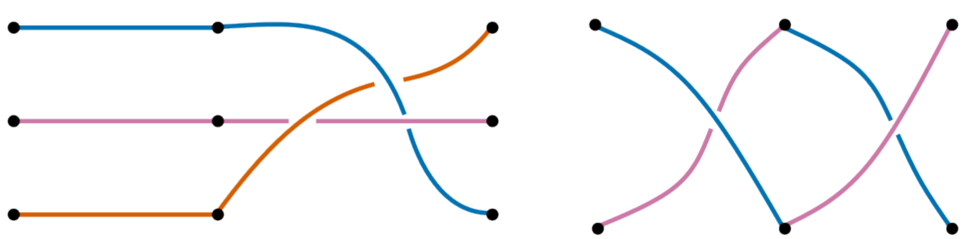
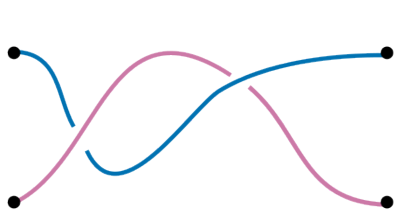


FIGURE 14. (a): On the left is the product of the 2 braids of Figure 9, denote it $e * \sigma$. Note that $e * \sigma = \sigma = \sigma * e$, since we can always pull the product strand back to $\sigma$. (b): On the right is the product of the braids of Figure 13. Note their product can be deformed to the identity, so they are inverses.

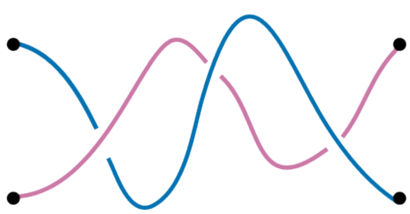
None of these identities or inverses of the last example (Example 4.9) would have worked without the liberty to pull strands around. Associativity similarly relies on this as well, since different parenthesis placement results in the component braids taking up different shares of space in their product. If the author's shady reasoning is to be believed, then we well and truly have a group!

**Example 4.10.** Figure 15 showcases more braid multiplication in $B_2$, just for fun!

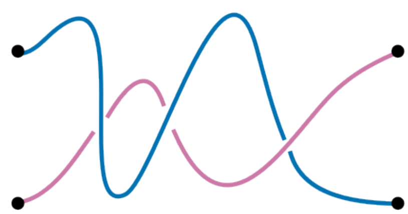


(A) $\sigma_1$

(B) $\sigma_2$



(C) $\sigma_1 * \sigma_2 = \sigma_2^{-1}$ by pulling up the pink through and pushing down the blue crest.



(D) $\sigma_2 * \sigma_1 = \sigma_2^{-1}$ by pulling up the blue through and pushing down the pink crest.

FIGURE 15. Braids in $B_2$.

**Example 4.11.** The braid group on 1 strand has just one element: the braid given by just a single straight line. Therefore, $B_1 \cong \{e\}$ where $\{e\}$ is the trivial group (Def. 2.4).

One can show the braid group on 2 strands is isomorphic to $(\mathbb{Z}, +)$.

**Example 4.12.** One thing we can observe about $B_n$ is that none of the braid groups for $n > 2$ are abelian. The strategy is simple: each braid $\{f_i : 1 \leq i \leq n\}$ has a corresponding permutation $\overline{f} \in S_n$. Then given two braids $\beta, \beta' \in B_n$ with permutations $\sigma, \sigma' \in S_n$, the permutation of $\beta * \beta'$ must be $\sigma' \circ \sigma$. Likewise, $\beta' * \beta$ must have permutation $\sigma \circ \sigma'$.

However, $S_n$ is not abelian for $n > 2$, so $B_n$ cannot be abelian for $n > 2$ as well. One neat way to see this is to recall the Dihedral group $D_3$ from Example 2.9, where from Figure 2, $F_3 \circ R \neq R \circ F_3$. But from the discussion in Definiton 2.11, elements of $D_3$ can be regarded as elements in $S_3$. Thus, take any braids $\beta, \beta' \in B_n$ that permute the first three points $p_1, p_2, p_3$ via $R$ and $F_3$ respectively (we can prove these exist simply by drawing an example). Then it cannot possibly be that $\beta * \beta' = \beta' * \beta$.

## 5. Mapping Class Groups

Now we have finally defined braid groups, but how do they relate to the symmetries of a disk? Like how we considered braids to be same up to "nudging," we now want to do the same for functions. Doing so will be a much more straightforward applications of the definitions from Section 3.

**Definition 5.1.** Let $U \subset \mathbb{R}^n$ be an arbitrary subset of $\mathbb{R}^n$, $V \subset U$ an arbitrary subset of $U$. We define

$$\text{Homeo}(U, V) = \{f : U \to U \mid f \text{ a homeomorphism such that } f(v) = v \text{ for all } v \in V\}.$$

In other words, $\text{Homeo}(U, V)$ is the set of all homeomorphisms of $U$ to itself that fix $V$ pointwise. Under composition, $\text{Homeo}(U, V)$ is a group.

**Remark 5.2.** As with homotopies, this definition extends to arbitrary topological spaces.

We will now use this group to define a relation.

**Definition 5.3.** Let $f, g \in \text{Homeo}(U, V)$ be arbitary. We say $f$ and $g$ are **isotopic in** $\text{Homeo}(U, V)$ if there is an isotopy

$$F : U \times [0, 1] \to U$$

from $f$ to $g$ such that for all $t \in [0, 1]$, $F_t \in \text{Homeo}(U, V)$.

Note that the use of isotopy rather than homotopy in the last definition (Def. 5.3) is unnecessary since the only allowed $F_t$ are homeomorphisms anyway. Nonetheless, it is standard to say isotopy.

Isotopy in $\text{Homeo}(U, V)$ defines an equivalence relation, which gives a partition of the set $\text{Homeo}(U, V)$ into subsets of functions that are isotopic in $\text{Homeo}(U, V)$. Like with braids, we would like to regard isotopic functions as equivalent and work with the equivalence classes. But do the resulting equivalence classes form another group?

The answer lies in the subset

$$\text{Homeo}_0(U, V) = \{f \in \text{Homeo}(U, V) : f \text{ is isotopic to } \text{id} \text{ in } \text{Homeo}(U, V)\}.$$

**Proposition 5.4.** *$\text{Homeo}_0(U, V)$ is a normal subgroup of $\text{Homeo}(U, V)$.*

*Proof.* First, we will show $\text{Homeo}_0(U, V)$ is a subgroup of $\text{Homeo}(U, V)$. Suppose $f, g \in \text{Homeo}_0(U, V)$. Then there exist isotopies $F, G : U \times [0, 1] \to U$ from $f$ and $g$ to the identity such that for all $t \in [0, 1]$, $F_t, G_t$ are homeomorphisms that fix $V$ pointwise. It follows that the function defined by

$$H : U \times [0, 1] \to U,$$
$$H(x, t) = F(G(x, t), t)$$

is a homotopy from $f \circ g$ to id such that for all $t \in [0, 1]$, $v \in V$, $H_t(v) = v$. Moreover, the function defined by

$$J : U \times [0, 1] \to U,$$
$$J(x, t) = F_t^{-1}(x)$$

is a homotopy from $f^{-1}$ to id that also fixes $V$ pointwise. It follows that $\text{Homeo}_0(U, V)$ is a subgroup.

Next, we will show $\mathrm{Homeo}_0(U,V) \leq \mathrm{Homeo}(U,V)$ is normal. Let $f \in \mathrm{Homeo}_0(U,V)$, $g \in \mathrm{Homeo}(U,V)$ be arbitrary. Then let $F : U \times [0,1] \to U$ be an isotopy of $f$ to the identity in $\mathrm{Homeo}(U,V)$. It follows that the function defined by

$$H : U \times [0,1] \to U,$$
$$H(x,t) = (g \circ F)(g^{-1}(x), t)$$

is a homotopy from $g \circ f \circ g^{-1}$ to id in $\mathrm{Homeo}(U,V)$. Thus, $g \circ f \circ g^{-1} \in \mathrm{Homeo}_0(U,V)$, so $\mathrm{Homeo}_0(U,V) \trianglelefteq \mathrm{Homeo}(U,V)$. □

We will now see why the normality of $\mathrm{Homeo}_0(U,V)$ is essential to creating our desired group.

**Proposition 5.5.** *Let $\sim$ be the relation on* $\mathrm{Homeo}(U,V)$ *where for all $f,g \in$* $\mathrm{Homeo}(U,V)$*, $f \sim g$ if and only if $f$ and $g$ are isotopic in* $\mathrm{Homeo}(U,V)$*. Then the equivalence classes of $\sim$ are exactly the cosets* $\mathrm{Homeo}_0(U,V)$ *in* $\mathrm{Homeo}(U,V)$*.*

*Proof.* It is sufficient to show that given any $f \in \mathrm{Homeo}(U,V)$,

$$[f]_\sim = f\,\mathrm{Homeo}_0(U,V).$$

Suppose $g \in [f]_\sim$. Then $g$ is isotopic to $f$ in $\mathrm{Homeo}(U,V)$. Let $F : U \times [0,1] \to U$ be an isotopy from $g$ to $f$ in $\mathrm{Homeo}(U,V)$. Then observe the function

$$H : U \times [0,1] \to U,$$
$$H(x,t) = (f^{-1} \circ F)(x,t)$$

is an isotopy from $f^{-1} \circ g$ to id in $\mathrm{Homeo}(U,V)$. This means that $f^{-1} \circ g \in \mathrm{Homeo}_0(U,V)$. But then

$$g = f \circ (f^{-1} \circ g) \in f\,\mathrm{Homeo}_0(U,V).$$

Conversely, suppose $g \in f\,\mathrm{Homeo}_0(U,V)$. Then $g = f \circ h$ where $h \in \mathrm{Homeo}_0(U,V)$. Let $F : U \times [0,1] \to U$ be an isotopy from $h$ to id in $\mathrm{Homeo}_0(U,V)$. It follows that the function

$$H : U \times [0,1] \to U,$$
$$H(x,t) = (f \circ F)(x,t)$$

is an isotopy from $f \circ h = g$ to $f$ in $\mathrm{Homeo}_0(U,V)$. Thus, $g \in [f]_\sim$. We conclude

$$[f]_\sim = f\,\mathrm{Homeo}_0(U,V).$$

Therefore, the equivalence classes $[f]_\sim$ are the same as the cosets of $\mathrm{Homeo}_0(U,V)$. □

This finally allows us to define our desired group of equivalence classes.

**Definition 5.6** (Mapping class groups)**.**

**Informal.** Recall how in the last section (Section 4), we grouped together all braids that were reasonably similar and considered them as one object. When we formed the braid groups, we operated on these classes of objects, and in fact this was necessary to reveal the group structure.

In this case, the situation is not quite as difficult. $\mathrm{Homeo}(U,V)$ is already a group. When our functions in $\mathrm{Homeo}(U,V)$ can be reasonably nudged to be the same, we want to regard them as equivalent and operate on those equivalence classes themselves.

In short, we want to turn the set of equivalence classes $[f]_\sim$ into a group. In this case, normal subgroups come to our rescue. The desired equivalence classes are, as luck would have it, precisely the cosets of a normal subgroup. The resulting quotient group

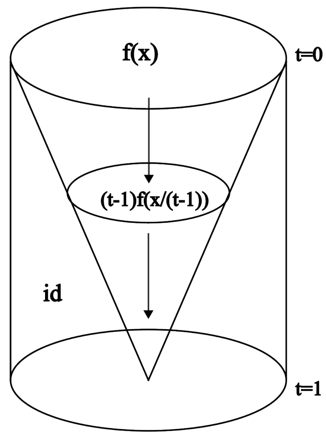



FIGURE 16. A standard visual of the Alexander trick by taking the function $(F(x,t),t)$ where $F$ is the isotopy. In other words, we are taking what $F(x,t)$ is at each time and moving that slightly out of the way of what it was before. No information is lost and we get a nice visualization.

has elements $[f]_\sim$ and inherits our desired group structure. We call this the *mapping class group* $MCG(U,V)$ and call the equivalence classes $[f]_\sim$ *mapping classes.*

**Formal.** The **mapping class group** of $U$ with respect to $V$ is the quotient group

$$MCG(U,V) = \text{Homeo}(U,V)/\,\text{Homeo}_0(U,V).$$

We will denote mapping classes simply as $[f]$, dropping the $\sim$. Very often, even the brackets are dropped, but we will avoid doing so in this paper.

We will work out a relevant example that will soon become very important: the mapping class group of the disk $D^2$ with respect to its boundary circle $S^1 \subset D^2$.

**Lemma 5.7** (Alexander trick [FM12, p. 47])**.** *The group $MCG(D^2,S^1)$ is trivial.*

*Proof.* Let $[f] \in MCG(D^2,S^1)$ be a mapping class. Then function $F : D^2 \times [0,1] \to D^2$ defined by

$$F(x,t) = \begin{cases} (1-t)f\left(\frac{x}{1-t}\right) & \text{if } 0 \le |x| < 1-t, \\ x & \text{if } 1-t \le |x| \le 1, \end{cases}$$

for all $t \in [0,1]$ is an isotopy of $f$ to id in $\text{Homeo}(D^2,S^1)$. □

The idea of the Alexander trick is that for every $t \in [0,1]$, we apply $f$ on a smaller disk of radius $1-t$. The factor of $1-t$ inside of $f$ allows us to map every point of $D^2$ for each $t$, while the factor of $1-t$ outside shrinks the image to another disk of radius $1-t$. While we shrink $f$ to a homeomorphism of smaller and smaller disks, we switch out the remaining space with the identity map. See Figure 16 for a visual.

However, the key point is this: every homeomorphism of the disk, so long as it fixes the boundary $S^1$, is homotopic to the identity. What this means is, like how we could interpret the function $r$ in Example 3.6 as squishing the cylinder, we can think of every boundary-fixing homeomorphism of the disk as some continuously pushing the points of the disk around. The pushing here is formally in accordance with the isotopy $F(x, 1-t)$ from id to $f$, where $F$ and $f$ are defined as in Lemma 5.7.

Intuitively, this makes such homeomorphisms of a disk rather intuitive: it is as if the disk were made of soft clay, and we are kneading it around while ensuring the whole disk remains covered, and without moving the boundary or breaking it.

This is not true in general for continuous functions or homeomorphisms. For example, even the homeomorphism $f : D^2 \to D^2$ given by $f(x) = -x$ cannot be interepreted this way.

**Remark 5.8.** For those who are interested, there is a lot to learn about mapping class groups. In particular, a lot of work has been done on the mapping class groups of surfaces, where for an orientable surface $S$, we focus on

$$MCG(S) = \mathrm{Homeo}^+(S, \partial S)/\,\mathrm{Homeo}_0(S, \partial S),$$

the mapping classes of orientation-preserving homeomorphisms. For example, we know that for every compact orientable surface, possibly with fintely many marked points that must be permuted, $MCG(S)$ is finitely presented [FM12, p. 137], and we can write a presentation down. The same is true for compact non-orientable surfaces with finitely many marked points [Kor02].

For orientable surfaces, [FM12] is great (I think it's amazing!). For non-orientable surfaces, try [Par14].

## 6. Punctured Disk and Braids

The group $MCG(D^2, S^1)$ was the final piece we needed to formally connect braids to disks.

**Theorem 6.1** ([FM12, p. 243])**.** *Let $D_n$ denote $D^2 \setminus \{p_1, \dots, p_n\}$. Assume none of the points $p_i$ are on the boundary circle of $D^2$. Then*

$$B_n \cong MCG(D_n, S^1).$$

This is not very elementary to prove, but the intuition is much easier. First, we must state the following characterization of homeomorphisms $D_n \to D_n$.

**Proposition 6.2.** *Fix $n$ points $p_1, \dots, p_n \in D^2$ not on the boundary. Let $\mathrm{Homeo}_p(D^2, S^1)$ denote the subgroup of $\mathrm{Homeo}(D^2, S^1)$ that permutes the $p_i$'s. In other words, for all $f \in \mathrm{Homeo}_p(D^2, S^1)$, there exists $\sigma \in S_n$ such that $f(p_i) = p_{\sigma(i)}$ for all $1 \le i \le n$. Then $\mathrm{Homeo}(D_n, S^1) \cong \mathrm{Homeo}_p(D^2, S^1)$.*

*Proof.*

**Informal.** The proof of this is a bit technical, but the idea is this. Whenever we have a homeomorphism of the punctured disk $D_n$, we can actually "fill in" the punctures and send them to each other. For any $f : D_n \to D_n$ a homeomorphism, $f$ being a homeomorphisms ensures each of these punctures has one and only one other puncture they can go to. We find this by checking all the points around each puncture. By continuity, they must all be sent to an area roughly around another puncture. This allows us to extend functions of punctured disks to functions of the whole disk that simply permute the $p_i$'s around.

This corresondence goes both ways. Given any homeomorphism that permutes the $p_i$'s, we can define a new homeomorphism $D_n \to D_n$ just by restricting the domain to $D_n$.

The proof is on the technical side. A rigorous understanding of it is not required.

**Formal.** Let $f \in \mathrm{Homeo}(D_n, S^1)$, and pick mutually disjoint punctured neighborhoods $U_i$ for each point $p_i$. We will define $\overline{f} \in S_n$ as follows. Since $f$ is a homeomorphism, the image of each $U_i$ must be another punctured neighborhood of some $p_j$, $1 \leq j \leq n$. Let $\overline{f}(i) = j$. Observe that $\overline{f}$ is indeed a bijection. If $i \neq j$, then the assumption $U_i \cap U_j = \varnothing$ implies $\overline{f}(U_i) \cap \overline{f}(U_j) = \varnothing$ because $f$ is a homeomorphism. But all punctured neighborhoods of the same puncture must intersect. Thus, $\overline{f}(i) \neq \overline{f}(j)$. It follows $\overline{f}$ is injective, hence bijective.

Then defining $\widetilde{f} : D^2 \to D^2$ by

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{if } x \in D_n \\ p_{\overline{f}(i)} & \text{if } x = p_i \end{cases},$$

we have a homeomorphism $\widetilde{f} \in \mathrm{Homeo}_p(D^2, S^1)$. This gives us a group homomorphism

$$\Phi : \mathrm{Homeo}(D_n, S^1) \to \mathrm{Homeo}_p(D^2, S^1),$$
$$\Phi(f) = \widetilde{f}.$$

We can define another homomorphism

$$\Psi : \mathrm{Homeo}_p(D^2, S^1) \to \mathrm{Homeo}(D_n, S^1)$$
$$\Psi(f) = f|_{D_n}$$

via restricting to $D_n$. $\Phi$ and $\Psi$ are inverses, so we conclude

$$\mathrm{Homeo}(D_n, S^1) \cong \mathrm{Homeo}_p(D^2, S^1).$$

□

Therefore, whenever we consider a homeomorphism $D_n \to D_n$, we may just as well consider a homeomorphism $D^2 \to D^2$ that permutes the $p_i$'s. This interpretation is quite useful — recall the discussion after the Alexander trick (Lemma 5.7). Any homeomorphism $D^2 \to D^2$ that fixes the boundary can be obtained by an isotopy in $\mathrm{Homeo}(D^2, S^1)$ from the identity.

This means that for any $f \in \mathrm{Homeo}(D_n, S^1)$, we can intuitively think of $f$ as corresonding to some "kneading" of $D^2$. To get $f$, we continuously move the points of $D^2$ around, gradually moving each point $x \in D^2$ to where the point $f(x)$ used to be. At the end, remove the $p_i$'s. Since we're thinking of deforming from the identity, this isotopy is the one from the Alexander trick, but backwards:

$$F(x,t) = \begin{cases} xf\left(\frac{x}{t}\right) & \text{if } 0 \leq |x| \leq t, \\ x & \text{if } t \leq |x| \leq 1. \end{cases}$$

Really all we are doing is reversing where $t = 0$ and $t = 1$ are on Figure 16.

But in this process, pay special attention to how each point $p_i$ traces out a path on $D^2$ to some $p_{\overline{f}(i)}$ as time moves from $t = 0$ to $t = 1$. There is the connection! For each $p_i$, we are getting a path

$$\gamma_i : [0,1] \to D^2$$

defined by how the point $p_i$ moves to $p_{\overline{f}(i)}$. We will now make one modification to the $\gamma_i$ by defining

$$f_i : [0,1] \to D^2 \times [0,1],$$
$$f_i(t) = (\gamma_i(t), t).$$

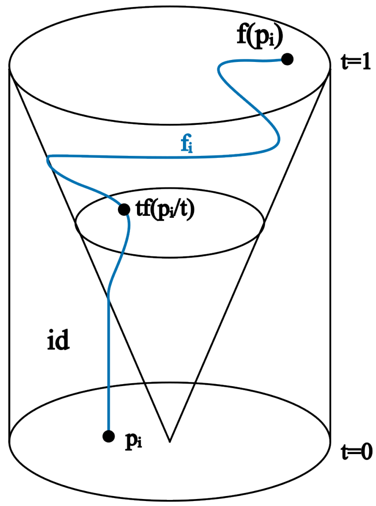



FIGURE 17. Observe how the Alexander lemma gives rise to the strands of a braid.

In effect, the $f_i$'s "raise" the paths outside of $D^2$, having them move forward as time passes. This is illustrated in Figure 17.

Observe that by definition, we have:

(i) the images $f_i[0,1]$ are disjoint. This is because if for any time $t$, $f_i(t) = f_j(t)$ for some $i \neq j$, then this implies that $p_i$ and $p_j$ were moved to the same location in the "kneading" process. As we have covered, this is not allowed.
Physically, this means the paths we have defined never intersect each other.
(ii) $f_i(0) = (\gamma_i(0), 0) = (p_i, 0)$.
(iii) $f_i(1) = (\gamma_i(1), 1) = (p_{\overline{f}(i)}, 1)$.
(iv) For any $t \in [0,1]$, $f_i(t) = (\gamma_i(t), t) \in D^2 \times \{t\}$.

But recall from Definition 4.1 that these conditions are precisely what defines a braid. Thus, from an arbitrary homeomorphism $f : D_n \to D_n$, we have obtained $\{f_i : 1 \leq i \leq n\}$, a braid on $n$ strands! Let's call $\{f_i : 1 \leq i \leq n\} = \beta_f$. But does this association still make sense for mapping classes?

Let's at least verify first that isotopic functions in $\mathrm{Homeo}(D_n, S^1)$ give equivalent braids.

**Proposition 6.3.** *The function*

$$\Phi : MCG(D_n, S^1) \to B_n$$

*defined by*

$$\Phi([f]) = \{f_i : 1 \leq i \leq n\} = \beta_f$$

*as in the preceding discussion is well-defined.*

*Proof.*

**Informal.** We wish to show that $\Phi$ makes sense. Of course, given an arbitrary homeomorphism in $\mathrm{Homeo}(D_n, S^1)$, it makes sense by our discussion that we just get the braid $\beta_f$. But for a quotient group like $MCG(D_n, S^1)$, if we take an arbitrary mapping class

$[f]$, do we get $\beta_f$ regardless of what representative we choose. It is not immediately clear that if $g \in [f]$, $\beta_g$ is equivalent to $\beta_f$.

The way we do it here is by working explicitly with the braid formulas for $f$ and $g$ given by the Alexander trick, then extending the isotopy $H$ from $f$ to $g$ to the entire braid. Visually, we are moving the whole braid from $f$ to $g$ in accordance with $H$. A bit more technical work is done to make this an ambient isotopy.

Ostensibly, this is the first step of a proof that $MCG(D_n, S^1) \cong B_n$. Using this strategy, we would want to show that $\Phi$ is also a homomorphism, and in fact bijective (technically, $\Phi$ is not a homomorphism in its current state, but we address this later). However, this is not how this theorem is conventionally proven. The proof is not easy, and we use much more powerful tools to tackle it. The following argument is included just to make the theorem seem more convincing using only elementary arguments. It may also demonstrate why continuing in this manner is not very sustainable.

**Formal.** Suppose $f \sim g$ in $\text{Homeo}(D_n, S^1)$. Then let $F, G : \mathbb{D} \to D^2$ be the isotopies from id to $F$ and $G$ respectively given by the Alexander trick. We use these to define $\widetilde{F}, \widetilde{G} : \mathbb{D} \to \mathbb{D}$,

$$\begin{aligned}\widetilde{F}(x,s) &= (F(x,s), s),\\ \widetilde{G}(x,s) &= (G(x,s), s).\end{aligned}$$

Since $f \sim g$, let $H : \mathbb{D} \to \mathbb{D}$ be an isotopy from id to $g \circ f^{-1}$ (given by filling in the missing points of the corresponding isotopy in $\text{Homeo}(D_n, S^1)$). Then $h : \mathbb{D} \times [0,1] \to \mathbb{D}$ defined by

$$h((x,s),t) = \begin{cases} \left(sH\left(\frac{x}{s}, t\right), s\right) & \text{if} 0 \le |x| \le s, \\ (x,s) & \text{if } s \le |x| \le 1. \end{cases}$$

satisfies

(i) for all $t \in [0,1]$, $h_t : \mathbb{D} \to \mathbb{D}$ is a homeomorphism,
(ii) for all $t \in [0,1]$, $h_t$ fixes the boundary of $\mathbb{D}$,
(iii) $h_0(x,s) = (x,s)$ is the identity and

$$\begin{aligned} h_1(f_i(s)) = h_1(F(x,s),s) &= \begin{cases} \left(s(g \circ f^{-1})\left(\frac{F(p_i,s)}{s}\right), s\right) & \text{if} 0 \le |F(p_i,s)| \le s, \\ (F(p_i,s),s) & \text{if } s \le |F(p_i,s)| \le 1 \end{cases} \\ &= \begin{cases} \left(s(g \circ f^{-1})\left(\frac{sf\left(\frac{p_i}{s}\right)}{s}\right), s\right) & \text{if } 0 \le |p_i| \le s, \\ (p_i,s) & \text{if } s \le |p_i| \le 1 \end{cases} \\ &= \begin{cases} \left(sg\left(\frac{p_i}{s}\right), s\right) & \text{if } 0 \le |p_i| \le s, \\ (p_i,s) & \text{if } s \le |p_i| \le 1 \end{cases} \\ &= \widetilde{G}(p_i,s) = g_i(s). \end{aligned}$$

It follows that $h$ is an ambient isotopy taking $\beta_f$ to $\beta_g$, so $\beta_f$ and $\beta_g$ are the same braid. Thus, $\Phi$ is independent of our choice of mapping class representative and therefore well-defined.

□

In the discussion preceding the proof of Proposition 6.3, we briefly mentioned $\Phi$ is not technically a homomorphism in its current state. This is because it's backwards:

$\Phi([f] \circ [g]) = \beta_g * \beta_f$. The reason is that $f \circ g$ is the function that applies $g$ first, then $f$; meanwhile $\beta_f * \beta_g$ is first the braid given by $f$, then the braid given by $g$.

The fix for this is easy. We just redefine composition in $MCG$ to go backwards. We will say $(f \circ g)(x) = g(f(x))$. This is not too unreasonable: if $\beta * \beta'$ is first $\beta$, then $\beta'$, then why should $f \circ g$ be $g$ first, then $f$? In some texts, this is the convention, and it ultimately makes no difference to the group structure (they are isomorphic). We will make this adjustment moving forward (I did not introduce it this way because it confuses me a lot).

The proof $\Phi$ is a homomorphism, and indeed an isomorphism, is much more involved. We will omit it in lieu of a more intuitive discussion.

**Author comment.** I am not actually aware if there is an elementary proof in this style. Probably there is a way to at least show $\Phi$ is a homomorphism, but I have not found or constructed an elementary argument for it using just ambient isotopy. If anyone has or knows of one, I would love to know it!

Ultimately, what does this isomorphism tell us about the relationship between braids and homeomorphisms of the punctured disk?

(i) First, the very fact $\Phi$ is a homomorphism tells us that if we deform the disk in accordance to $f$, then in accordance to $g$, that very process is no different (up to isotopy by homeomorphisms) to deforming the disk in accordance to $f \circ g$.
(ii) Second, that $\Phi$ is a bijective tells us that every braid arises from a mapping class of homeomorphisms in $\text{Homeo}(D_n, S^1)$. Likewise, every mapping class of homeomorphisms in $\text{Homeo}(D_n, S^1)$ uniquely gives a braid — any other mapping class will give a different braid.
(iii) Third, from the correspondence of mapping classes and braids, we see that kneading around a homeomorphism in $\text{Homeo}(D_n, S^1)$ is essentially no different from nudging around a braid on $n$ strands.

This brings us to our final point: by endowing group structures to our objects — homeomorphisms and braids — we are able to speak precisely about the correspondence of structures between homeomorphisms and braids. Braids and mapping classes in $MCG(D_n, S^1)$ correspond one-to-one, and the natural operations on each correspond perfectly as well. Up to multiplying, these are exactly the same.

Besides just being interesting, this correspondence is quite useful for talking about mapping classes in general. For example, while homeomorphisms can be rather difficult to write down explcitly, in the case of the disk, we can simply specify a braid rather than write down an explicit formula. Writing down and composing functions can all be substituted by drawing braids. Or we can tell if two functions are the same by examining their braids. and when we compose functions, we need only multiply their braids.

This is not limited to the disk. "Surfaces," such as the sphere, torus, klein bottle, etc. all contain disks. If a torus has some punctures in it, we can cut out a disk, apply braids, then glue that disk back to get a mapping class of the torus (we call this a *half dehn twist*).

This means, remarkably, that braids do not just describe the mapping classes of a punctured disk, but in fact of all shapes that have disks inside of them. All because we realized braids can be multiplied.

## 7. The proof?

The main storyline of this paper is over, but probably there are readers interested in the proof that $MCG(D_n, S^1) \cong B_n$.

The techniques are quite beyond the scope of this paper, but they are worth learning. They are accessible after a typical first course in algebraic topology.

The braid group on $n$ strands is the same as the fundamental group of the unordered configuration space $C(D^2, n)$. Then we use the generalized *Birman exact sequence*, where for any orientable surface $S$, $S_n = S \setminus \{p_1, \ldots, p_n\}$, there is an exact sequence

$$1 \to \pi_1(C(S,n)) \xrightarrow{\text{Push}} MCG(S_n, \partial S_n) \xrightarrow{\text{Forget}} MCG(S, \partial S) \to 1.$$

Roughly, we define Push by taking *Dehn twists* around the loops in $C(S,n)$, Forget is the natural map where we fill in the punctures, and the exact sequence is obtained by the long exact sequence of homotopy groups of the fiber bundle (the LES of a fiber bundle is covered in Chapter 4 of [Hat02]),

$$\text{Homeo}^+(S_n, \partial S_n) \to \text{Homeo}^+(S, \partial S) \to C(\text{int}(S), n).$$

Of course, since $MCG(D^2, S^1)$ is trivial, we get

$$B_n \cong \pi_1(C(D^2, n)) \cong MCG(D^2, S^1).$$

The full exposition is covered in detail in [FM12].

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