BU Society of Mathematics Colloquium Talk: The Lattice Isomorphism Theorem and an Introduction to Correspondences

Molly Roughan

February 20, 2026

Abstract

Quotient constructions appear throughout algebra. In linear algebra, they arise from kernels of linear maps. In group theory, they arise from normal subgroups. The Lattice Isomorphism Theorem (also called the fourth Isomorphism Theorem) describes how subgroup structure behaves under quotients. This talk builds upon existing intuition for the theorem, illustrates it through examples in $\mathbb{Z}$ and the dihedral groups, and concludes with some interesting applications of the theorem.

1 Linear Algebra Analogy

Before we talk about groups, I want to start somewhere familiar. I often like to say "all roads lead back to linear algebra," so I will start there. In linear algebra, we can take quotients of vector spaces. If we have a linear map $T : V \to W$, then we form the quotient $V/\ker(T)$. By taking this linear map, we "collapse" the kernel, so everything that gets sent to zero by T is identified with zero in the quotient. The *quotient space* is then the information that survives.

Definition 1. *Let $T : V \to W$ be a linear transformation. The kernel is*

$$\ker(T) = \{v \in V \mid T(v) = 0\}.$$

The quotient space $V/\ker(T)$ identifies $v_a \sim v_b$ whenever $(v_a - v_b) \in \ker(T)$.

Essentially, if v_a and v_b differ by something in the kernel (e.g., something that gets sent to 0) under T, then v_a and v_b are the same in the quotient.

Moving onto groups, we can ask the same question: What survives when we "collapse" a normal subgroup?

2 What is a Lattice?

2.1 Group Theory Refresher

Definition 2. *Let $\varphi : G \to H$ be a group homomorphism. The kernel of φ is*

$$\ker(\varphi) = \{g \in G \mid \varphi(g) = e\}.$$

This definition looks almost identical to the linear algebra one. Again, the kernel consists of everything that is identified with the identity, e, under the map.

Definition 3. *A subgroup $N \subseteq G$ is normal, written $N \trianglelefteq G$, if*

$$gNg^{-1} = N \quad \textit{for all } g \in G.$$

It can be most simply thought of, as one may recall from the last talk, of being sets where everything behaves as one would like.

If $N \trianglelefteq G$, we define the quotient group G/N, whose elements are in the form

$$gN = \{gn \mid n \in N\}.$$

2.2 Subgroup Lattices

Subgroup lattices are visual representations of subgroups; they are structures where the higher-up subgroups contain all subgroups below them. Higher nodes represent larger subgroups and lower nodes represent smaller subgroups.

Example 1. *Subgroups of $\mathbb{Z}$ are precisely $k\mathbb{Z}$, which makes them easy to work with. For a simple example, consider the subgroups containing $12\mathbb{Z}$, These are: $\mathbb{Z}$, $2\mathbb{Z}$, $3\mathbb{Z}$, $4\mathbb{Z}$, $6\mathbb{Z}$, and $12\mathbb{Z}$.*

$$\begin{tikzcd}
& & \mathbb{Z} = \langle 1 \rangle \arrow[dl, dash] \arrow[dr, dash] & \\
& \langle 2 \rangle \arrow[dl, dash] \arrow[dr, dash] & & \langle 3 \rangle \arrow[dl, dash] \\
\langle 4 \rangle \arrow[dr, dash] & & \langle 6 \rangle \arrow[dl, dash] & \\
& 12\mathbb{Z} = \langle 12 \rangle & &
\end{tikzcd}$$

Up until now, $\mathbb{Z}$ has been a very friendly example because it is abelian, so everything commutes and every subgroup is normal. An interesting non-abelian example is the dihedral group of a square, denoted D_8 because it has 8 elements. Geometrically, it is the group of all symmetries of a square: four rotations (r) and four reflections (s). (Moreover, any dihedral group has 2n elements, for an n-sided polygon.)

Definition 4. *Define*

$$D_8 = \langle r, s \mid r^4 = e,\ s^2 = e,\ srs = r^{-1} \rangle.$$

Here, r represents rotation by 90° and s represents a reflection.

Example 2. *Why not all subgroups of D_8 are normal: If you rotate a square and then reflect it, that is the same as reflecting first and then rotating the **opposite** direction direction, not the first direction.*

The rotation subgroup $\langle r \rangle = \{e, r, r^2, r^3\}$ is normal in D_8, but the the reflection subgroup $\langle s \rangle = \{e, s\}$ is not. A quick way to see this is that conjugating s by a rotation produces a different reflection. Geometrically, a 90° rotation changes the axis of reflection.

$$\begin{tikzcd}
 & & D_8 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & & \\
 & \langle s, r^2 \rangle \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \langle r \rangle \arrow[d, dash] & \langle rs, r^2 \rangle \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\langle s \rangle \arrow[drr, dash] & \langle r^2 s \rangle \arrow[dr, dash] & \langle r^2 \rangle \arrow[d, dash] & \langle rs \rangle \arrow[dl, dash] & \langle r^3 s \rangle \arrow[dll, dash] \\
 & & \{e\} & &
\end{tikzcd}$$

Later, when we apply the correspondence theorem, we will quotient by $N = \langle r^2 \rangle$ and have a great visual example for the theorem.

3 The Lattice Isomorphism Theorem

The Lattice Isomorphism Theorem, also called the Correspondence Theorem, is the fourth isomorphism theorem and defined as such:

Theorem 1 (Lattice Isomorphism Theorem)**.** *Let $N \trianglelefteq G$. Then there is a bijection between subgroups of G/N and the subgroups A of G satisfying $N \subseteq A \subseteq G$ via the correspondence $A \mapsto A/N$.*

3.1 Intuition

If $N \trianglelefteq G$, then every subgroup of G/N looks like A/N for some subgroup A with $N \subseteq A \subseteq G$. This correspondence essentially shrinks and relabels the quotient.

Theorem 2. *Under the correspondence $A \mapsto A/N$, $A \trianglelefteq G \leftrightarrow A/N \trianglelefteq G/N$, provided $N \subseteq A$.*

This is the formal way to write that normality is preserved and reflected, so if a subgroup was stable under conjugation previously, its image is stable as well. If a subgroup was not normal, the quotient does not fix that, thus, the quotient preserves symmetry as well.

3.2 Examples

Example 3. $G = \mathbb{Z},\ N = 12\mathbb{Z}$

Take a look at the lattice for $G/N = \mathbb{Z}/12\mathbb{Z}$ from Example 1, and notice it's similarities to the earlier lattice for subgroups of G containing N.

$$\begin{tikzcd}
 & & \mathbb{Z}/12\mathbb{Z} \arrow[dl, dash] \arrow[dr, dash] & \\
 & \langle 2 \rangle \arrow[dl, dash] \arrow[dr, dash] & & \langle 3 \rangle \arrow[dl, dash] \\
\langle 4 \rangle \arrow[dr, dash] & & \langle 6 \rangle \arrow[dl, dash] & \\
 & \{0\} & &
\end{tikzcd}$$

Overlaying these diagrams gives a very clear visual example of the Lattice Isomorphism Theorem.

$$\begin{tikzcd}
 & & \mathbb{Z} \mapsto \mathbb{Z}/12\mathbb{Z} \arrow[dl, equal] \arrow[dr, equal] & \\
 & \langle 2 \rangle \arrow[dl, equal] \arrow[dr, equal] & & \langle 3 \rangle \arrow[dl, equal] \\
\langle 4 \rangle \arrow[dr, equal] & & \langle 6 \rangle \arrow[dl, equal] & \\
 & 12\mathbb{Z} \mapsto \{0\} & &
\end{tikzcd}$$

Example 4. *Let $G = D_8$ and $N = \langle r^2 \rangle$.*

Recall the diagram from Example 2 for subgroups of $G = D_8$ containing $\langle r^2 \rangle$ and compare it to the below diagram for subgroups of G/N.

$$\begin{tikzcd}
 & D_8/\langle r^2 \rangle \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\langle s, r^2 \rangle/\langle r^2 \rangle \arrow[dr, dash] & \langle r \rangle/\langle r^2 \rangle \arrow[d, dash] & \langle rs, r^2 \rangle/\langle r^2 \rangle \arrow[dl, dash] \\
 & \{e\} = \langle r^2 \rangle/\langle r^2 \rangle &
\end{tikzcd}$$

Subgroups of D_8 containing $\langle r^2 \rangle$ correspond exactly to subgroups of $D_8/\langle r^2 \rangle$. Moreover, normal subgroups correspond to normal subgroups.

$$
\begin{tikzcd}
 & & D_8 \mapsto D_8/\langle r^2\rangle \arrow[dl, equal] \arrow[d, equal] \arrow[dr, equal] & & \\
 & \langle s, r^2\rangle \arrow[dl, dash] \arrow[d, dash] \arrow[dr, equal] & \langle r\rangle \arrow[d, equal] & \langle rs, r^2\rangle \arrow[dl, equal] \arrow[d, dash] \arrow[dr, dash] & \\
\langle s\rangle \arrow[drr, dash] & \langle r^2 s\rangle \arrow[dr, dash] & \langle r^2\rangle \mapsto \{e\} \arrow[d, dash] & \langle rs\rangle \arrow[dl, dash] & \langle r^3 s\rangle \arrow[dll, dash] \\
 & & \{e\} & &
\end{tikzcd}
$$

3.3 Formality and Proof

Let $N \trianglelefteq G$ and define the projection map

$$\pi : G \to G/N, \qquad \pi(g) = gN.$$

Notice that $\ker(\pi) = N$.

1. Subgroups containing N map to subgroups of G/N. If A is a subgroup of G with $N \subseteq A$, then
$$A/N = \{aN \mid a \in A\}$$
is a subgroup of G/N.

2. Subgroups of G/N lift to subgroups containing N. If $H \leq G/N$, define
$$\pi^{-1}(H) = \{g \in G \mid \pi(g) \in H\}$$
. Then $\pi^{-1}(H)$ is a subgroup of G and satisfies $N \subseteq \pi^{-1}(H)$.

3. These constructions undo each other. If $H \leq G/N$, then
$$\pi(\pi^{-1}(H)) = H$$
. If $N \subseteq A \subseteq G$, then
$$\pi^{-1}(A/N) = A.$$
Thus the assignments
$$A \mapsto A/N \text{ and } H \mapsto \pi^{-1}(H)$$
are inverse constructions. Therefore, there is a bijection between subgroups of G/N and subgroups of G containing N.

4 Uses and Interesting Results

4.1 Prime Order and Maximal N

Theorem 3. *A normal subgroup $N \trianglelefteq G$ is maximal if and only if $|G/N|$ is prime.*

Proof. By the Lattice Isomorphism Theorem, for some H such that $N \subseteq H \subseteq G$, $H \mapsto$ subgroups of G/N. If N is maximal, however, there are no such H, so there can be no other non-trivial subgroup besides G/N itself. *(Sounds like prime numbers, no?)*

Indeed, if N is maximal, the $|G/N|$ is prime. Conversely, if $|G/N|$ is prime, then G/N has only the trivial subgroup and itself. □

Returning to where we began: in linear algebra, subspaces of $V/\ker(T)$ correspond to subspaces of V containing $\ker(T)$. If $\dim(V/\ker(T)) = 1$, there are no intermediate subspaces. This is very similar to prime order in groups.

4.2 Lattices of Form $\mathbb{Z}/p^n\mathbb{Z}$

An interesting pattern of lattices that appears is that in the form of $G = \mathbb{Z}$, $N = p^n\mathbb{Z}$ for prime p and integer n.

Because $\mathbb{Z}/p^n\mathbb{Z}$ is cyclic and of order p^n, it has exactly one subgroup of order p^k for each k, which itself has the same property. Thus, the lattice simply forms a chain. Similarly, by the Lattice Isomorphism Theorem, the subgroups of G containing N will match this chain structure.

$$\begin{tikzcd}
\mathbb{Z}/p^n\mathbb{Z} \arrow[d, dash] \\
\langle p \rangle \arrow[d, dash] \\
\langle p^2 \rangle \arrow[d, dash] \\
\langle p^3 \rangle \arrow[d, dash] \\
\vdots \arrow[d, dash] \\
\langle p^{n-1} \rangle \arrow[d, dash] \\
\langle p^n \rangle = \{0\}
\end{tikzcd}$$

4.3 Extensions

This talk has not covered it much, but the Lattice Isomorphism Theorem is broadly applicable to other areas of math. As touched on in section 4.1, it can be brought back to Linear Algebra, but also extended to rings (and further). For example, if I is an ideal in a ring R, then ideals of R/I correspond to ideals of R containing I, and many of the same properties that can be garnered from the Lattice Isomorphism Theorem are preserved.